HSC Extension 2 2006 Solution
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Question 1

(a) $\int \frac{x}{\sqrt{9-4x^2}}\,dx = -\frac{1}{8}\int \frac{-8x}{\sqrt{9-4x^2}}\,dx$

$= -\frac{1}{8}.2\sqrt{9-4x^2} + C$

$= -\frac{\sqrt{9-4x^2}}{4} + C.$

(b) $\int \frac{dx}{x^2-6x+13} = \int \frac{1}{(x-3)^2+4}\,dx$

$= \frac{1}{2}\tan^{-1}\frac{x-3}{2} + C.$

(c) $a = \lim_{x\to 3}\frac{16x-43}{x+2} = \frac{5}{5} = 1, c = \lim_{x\to -2}\frac{16x-43}{(x-3)^2} = \frac{-75}{25} = -3$

$b = 3$, by equating the coefficients of x^2.

$\int \frac{16x-43}{(x-3)^2(x+2)}\,dx = \int\left(\frac{1}{(x-3)^2} + \frac{3}{(x-3)} - \frac{3}{(x+2)}\right)dx$

$= \frac{-1}{x-3} + 3\ln(x-3) - 3\ln(x+2) + C.$

(d) Let $u = t, dv = e^{-t}dt$ then $du = dt, v = -e^{-t}$.

$\int_0^2 te^{-t}\,dt = \left[-te^{-t}\right]_0^2 + \int_0^2 e^{-t}\,dt$

$= -2e^{-2} - \left[e^{-t}\right]_0^2$

$= -2e^{-2} - e^{-2} + 1$

$= -3e^{-2} + 1.$

(e) Let $t = \tan\frac{\theta}{2}, dt = \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta, \therefore d\theta = \frac{2dt}{1+t^2}.$

When $\theta = \frac{\pi}{2}, t = 1$. When $\theta = \frac{2\pi}{3}, t = \tan\frac{\pi}{3} = \sqrt{3}.$

$\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{1}{\sin\theta}\,dx = \int_1^{\sqrt{3}} \frac{1+t^2}{2t}\,\frac{2dt}{1+t^2}$

$= \int_1^{\sqrt{3}} \frac{1}{t}\,dt$

$= [\ln t]_1^{\sqrt{3}}$

$= \ln\sqrt{3}.$

Question 2

(a) (i) $z^2 = (3+i)^2 = 9-1+6i = 8+6i.$

(ii) $\bar{z}w = (3-i)(2-5i) = 6-5-2i-15i = 1-17i.$

(iii) $\frac{w}{z} = \frac{2-5i}{3+i} = \frac{(2-5i)(3-i)}{(3+i)(3-i)} = \frac{1-17i}{10}.$

(b) (i) $\sqrt{3} - i = 2\,\text{cis}\left(-\frac{\pi}{6}\right).$

(ii) $\left(\sqrt{3}-i\right)^7 = 128\,\text{cis}\frac{-7\pi}{6}$

$= 128\,\text{cis}\frac{5\pi}{6}$

(iii) $\left(\sqrt{3}-i\right)^7 = 64(-\sqrt{3}+i).$

(c) $-1 = \text{cis}(\pi + 2k\pi).$

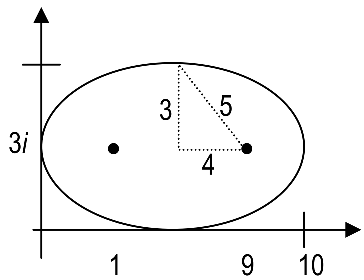
$z = \sqrt[3]{-1} = \text{cis}\frac{\pi+2k\pi}{3}, k = 0, \pm 1,$

$= \text{cis}\frac{\pi}{3}, \text{cis}\left(-\frac{\pi}{3}\right), \text{cis}\,\pi = -1.$

(d) (i) $5+3i.$

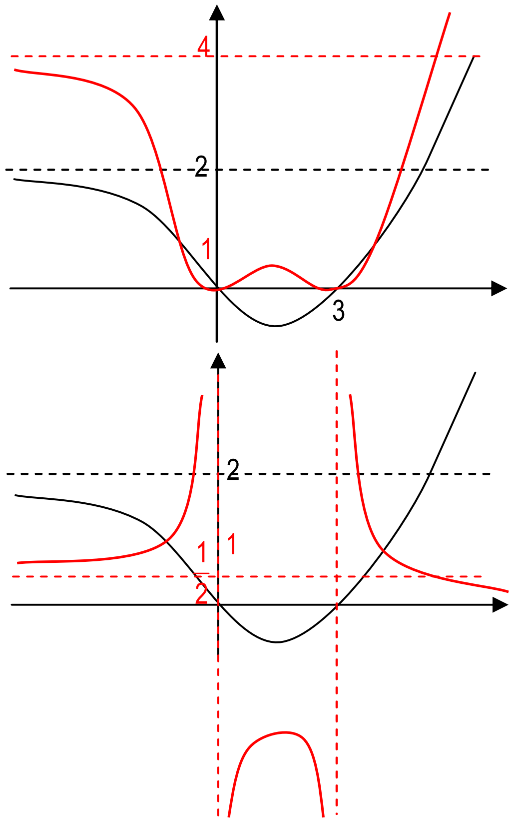
(ii) major axis $= 10$, minor axis $= 6$.

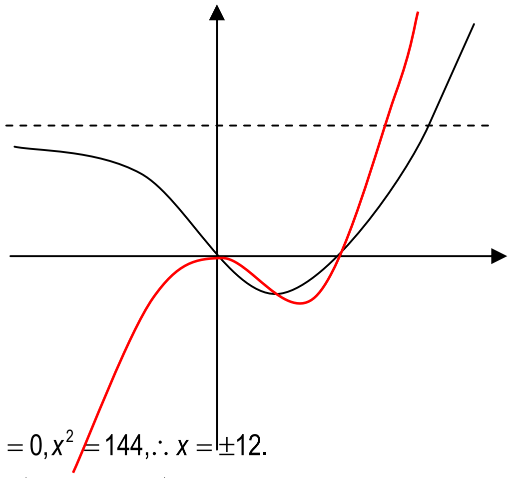
(iii) $0 \le \arg(z) \le \frac{\pi}{2}.$



Question 3

(a)





(b) (i) $y=0, x^2=144, \therefore x=\pm 12.$

(ii) $\pm\left(\sqrt{144+25},0\right)=\pm(13,0).$

(iii) directrices $x=\pm\dfrac{144}{13}.$

asymptotes $y=\pm\dfrac{5x}{12}.$

(c) (i) $\sum\alpha=4a=12, \therefore a=3.$

$\prod\alpha=\left(a^2+b^2\right)\left(a^2+4b^2\right)=130.$

$\left(9+b^2\right)\left(9+4b^2\right)=130.$

$4b^4+45b^2-49=0.$

$(4b^2+49)(b^2-1)=0.$

$\therefore b^2=1$ (b is real)

$\therefore b=1.$

(ii) $P(x)=(x-3+i)(x-3-i)(x-3+2i)(x-3-2i)$

$=\left((x-3)^2+1\right)\left((x-3)^2+4\right)$

$=\left(x^2-6x+10\right)\left(x^2-6x+13\right).$

Question 4

(a) $p'(x)=3ax^2+b.$

$p'(1)=0$ gives $3a+b=0.$ (1)

$p(1)=0$ gives $a+b+c=0.$ (2)

$p(-1)=4$ gives $-a-b+c=4.$ (3)

$(2)+(3)$ gives $2c=4, \therefore c=2.$

$(1)+(3)$ gives $2a+2=4, \therefore a=1.$

(1) gives $b=-3.$

(b) Side $=2x.$

Area $=4x^2.$

$\partial V=4x^2\partial y.$

$V=\int_0^1 4x^2\,dy=\int_0^1 4y\,dy$

$=\left[2y^2\right]_0^1=2$ units^3.

(c) (i) $m_1=\dfrac{\frac{1}{p}-\frac{1}{q}}{p-q}=-\dfrac{1}{pq}, \therefore m_2=pq.$

$y-\dfrac{1}{r}=pq(x-r)$

$y=pqx-pqr+\dfrac{1}{r}.$ (1)

(ii) $y=qrx-pqr+\dfrac{1}{p}.$ (2)

(iii) $(1)-(2)$ gives

$0=q(p-r)x+\dfrac{1}{r}-\dfrac{1}{p}.$

$0=q(p-r)x+\dfrac{p-r}{pr}$

$0=qx+\dfrac{1}{pr}.$

$\therefore x=-\dfrac{1}{pqr}.$

Put to (1), $y=pqx-pqr+\dfrac{1}{r}$

$=-\dfrac{1}{r}-pqr+\dfrac{1}{r}$

$=-pqr.$

Since $xy=\left(-\dfrac{1}{pqr}\right)(-pqr)=1,$ this point belongs to the hyperbola.

(d) (i) $KM\,//\,BL$ and $KM=BL$ (the join of the midpoints of two sides of a triangle is parallel to and equal half the third side.)

(ii) the interior angle in cyclic quad $KMLP$ is equal to the opposite exterior angle. Let $\angle KML$ be α.

(ii) $KB=KA$ (K is midpoint of AB)

$KP=KB$ ($\angle KBP=\angle KPB=\alpha, \therefore$ isos Δ)

$\therefore KP=KA.$

$\therefore \angle KPA=\angle KAP$. Let it be β.

$\therefore \angle KPB+\angle KPA=\alpha+\beta.$

But $2\alpha+2\beta=180°$(angle sum in ΔABP)

$\therefore \angle KPB+\angle KPA=90°, \therefore AP\perp BP.$

Question 5

(a) $\partial V=2\pi xy\partial x$

$V=2\pi\int_0^1 x^2(x-1)^2\,dx$

$=2\pi\int_0^1\left(x^4-2x^3+x^2\right)dx$

$=2\pi\left[\dfrac{x^5}{5}-\dfrac{2x^4}{4}+\dfrac{x^3}{3}\right]_0^1$

$=\dfrac{\pi}{15}$ units^3.

(b) (i) $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$.   (1)

$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$.   (2)

$(1)+(2)$ gives

$\cos(\alpha+\beta)+\cos(\alpha-\beta)=2\cos\alpha\cos\beta$.

(ii) $\cos\theta+\cos 3\theta+\cos 2\theta+\cos 4\theta$

$=2\cos\theta\cos 2\theta+2\cos\theta\cos 3\theta$

$=2\cos\theta\left(\cos 2\theta+\cos 3\theta\right)$

$=2\cos\theta.2\cos\frac{\theta}{2}\cos\frac{5\theta}{2}$.

Solve $2\cos\theta.2\cos\frac{\theta}{2}\cos\frac{5\theta}{2}=0$ gives

$\cos\theta=0,\cos\frac{\theta}{2}=0,\cos\frac{5\theta}{2}=0$.

$\therefore\theta=\frac{\pi}{2}+k\pi,\frac{\theta}{2}=\frac{\pi}{2}+k\pi,\frac{5\theta}{2}=\frac{\pi}{2}+k\pi$.

$\therefore\theta=\frac{\pi}{2}+k\pi,\theta=\pi+2k\pi,\theta=\frac{\pi}{5}+\frac{2k\pi}{5}$.

$\therefore\theta=\frac{\pi}{2},\frac{3\pi}{2},\pi,\frac{\pi}{5},\frac{3\pi}{5},\frac{7\pi}{5},\frac{9\pi}{5}$.

(c) (i) vert, $T_1\cos\alpha=T_2\cos\alpha+mg$.   (1)

hor, $T_1\sin\alpha+T_2\sin\alpha=m\ell\sin\alpha\omega^2$.   (2)

(ii) Put $T_2=0$,

$\frac{(2)}{(1)}$ gives $\tan\alpha=\frac{m\ell\sin\alpha\omega^2}{mg}$.

$\therefore\omega^2=\frac{g}{\ell\cos\alpha}$.

(d) (i) 3^4.

(ii) $0.2\times0.6\times0.2\times0.6=0.0144$.

(iii) Pr = 4W + 3W + 2W2D + 2W1D1L + 1W3D

$=0.2^4+{}^4C_3 0.2^3\times0.8+\frac{4!}{2!2!}0.2^2\times0.6^2$

$+\frac{4!}{2!}0.2^2\times0.6\times0.2+\frac{4!}{3!}0.6^3\times0.2$

$=0.344$.

Question 6

(a) (i) Using the Sine rule,

$\frac{\sin\theta}{OB}=\frac{\sin(\beta-\theta)}{OA}$.

$\therefore\frac{OA}{OB}=\frac{\sin(\beta-\theta)}{\sin\theta}$.

(ii) Similarly, $\frac{OB}{OC}=\frac{\sin(\gamma-\theta)}{\sin\theta},\frac{OC}{OA}=\frac{\sin(\alpha-\theta)}{\sin\theta}$.

Multiplying all the three,

$\frac{OA}{OB}.\frac{OB}{OC}.\frac{OC}{OA}=\frac{\sin(\beta-\theta)}{\sin\theta}.\frac{\sin(\gamma-\theta)}{\sin\theta}.\frac{\sin(\alpha-\theta)}{\sin\theta}$.

$\therefore\sin^3\theta=\sin(\beta-\theta)\sin(\gamma-\theta)\sin(\beta-\theta)$.

(iii) $\cot x-\cot y=\frac{\cos x}{\sin x}-\frac{\cos y}{\sin y}$

$=\frac{\sin y\cos x-\cos y\sin x}{\sin x\sin y}$

$=\frac{\sin(y-x)}{\sin x\sin y}$.

(iv) $\left(\cot\theta-\cot\alpha\right)\left(\cot\theta-\cot\beta\right)\left(\cot\theta-\cot\gamma\right)$

$=\frac{\sin(\theta-\alpha)}{\sin\theta\sin\alpha}\frac{\sin(\theta-\beta)}{\sin\theta\sin\beta}\frac{\sin(\theta-\gamma)}{\sin\theta\sin\gamma}$

$=\frac{\sin^3\theta}{\sin\theta\sin\alpha\sin\theta\sin\beta\sin\theta\sin\gamma}$, from (ii)

$=\operatorname{cosec}\alpha\operatorname{cosec}\beta\operatorname{cosec}\gamma$.

(v) If $\alpha=\frac{\pi}{2},\beta=\gamma=\frac{\pi}{4}$,

$\left(\cot\theta-\cot\frac{\pi}{2}\right)\left(\cot\theta-\cot\frac{\pi}{4}\right)\left(\cot\theta-\cot\frac{\pi}{4}\right)$

$=\operatorname{cosec}\frac{\pi}{2}\operatorname{cosec}\frac{\pi}{4}\operatorname{cosec}\frac{\pi}{4}$.

$\cot\theta\left(\cot\theta-1\right)^2=2$.

$\cot^3\theta-2\cot^2\theta+\cot\theta-2=0$.

$(\cot\theta-2)(\cot^2\theta)+\cot\theta-2=0$

$(\cot\theta-2)(\cot^2\theta+1)=0$.

$\cot\theta=2$.

$\theta=\tan^{-1}\frac{1}{2}$.

(b) (i) On the planet's surface, $\ddot{x}=-g$.

$-g=-\frac{k}{R^3}$.

$\therefore k=gR^3$.

(ii) $\ddot{x}=\frac{vdv}{dx}=-\frac{gR^3}{x^3}$.

$\int v\,dv=-gR^3\int\frac{1}{x^3}\,dx$

$\frac{v^2}{2}=gR^3.\frac{1}{2x^2}+C$.

When $x=R,v=u,\therefore C=\frac{u^2}{2}-\frac{gR}{2}$.

$\therefore v^2=\frac{gR^3}{x^2}+u^2-gR$.

$=\frac{gR^3}{x^2}-\left(gR-u^2\right)$.

(iii) If $u^2\geq gR$ then $v^2\geq\frac{gR^3}{x^2}$.

$v^2\geq\frac{gR^3}{R^2+3uRt-(gR-u^2)t^2}$.

As $t \to \infty, v^2 \to 0$: the particle will not return.

(iv) (1) When $v = 0, \dfrac{gR^3}{x^2} = (gR - u^2)$,

$$\therefore x^2 = \frac{gR^3}{gR - u^2}.$$

$$\therefore D = \sqrt{\frac{gR^3}{gR - u^2}}.$$

(2) $\left(\dfrac{gR^3}{gR - u^2}\right) = R^2 + 2uRt - (gR - u^2)t^2$

$$(gR - u^2)t^2 - 2uRt + R^2 - \left(\frac{gR^3}{gR - u^2}\right) = 0.$$

$$t = \frac{uR + \sqrt{u^2R^2 - (gR - u^2)\left(R^2 - \left(\frac{gR^3}{gR - u^2}\right)\right)}}{(gR - u^2)}$$

$$= \frac{uR}{(gR - u^2)}.$$

Question 7

(a) (i) $\cos x = \tan x$

$$= \frac{\sin x}{\cos x}.$$

$\cos^2 x - \sin x = 0.$

If $x = \alpha$ is the point of intersection, then

$\cos^2 \alpha - \sin\alpha = 0.$ (1)

For $y = \cos x, \dfrac{dy}{dx} = -\sin x. \therefore m_1 = -\sin\alpha.$

For $y = \tan x, \dfrac{dy}{dx} = \sec^2 x. \therefore m_2 = \sec^2 \alpha = \dfrac{1}{\cos^2 \alpha}.$

But $\cos^2 \alpha = \sin\alpha.$

$$\therefore m_1 m_2 = -\sin\alpha . \frac{1}{\cos^2 \alpha} = -1.$$

$\therefore$ The curves intersect at right angles at $x = \alpha$.

(ii) From (1),

$1 - \sin^2 \alpha - \sin\alpha = 0.$

$\sin^2 \alpha + \sin\alpha - 1 = 0.$

$\sin\alpha = \dfrac{-1+\sqrt{5}}{2}$ (taking the + value for $0 < \alpha < \dfrac{\pi}{2}$.)

$$\tan\alpha = \frac{-1+\sqrt{5}}{\sqrt{4 - (-1+\sqrt{5})^2}}$$

$$= \frac{-1+\sqrt{5}}{\sqrt{4 - 1 + 2\sqrt{5} - 5}}$$

$$= \frac{-1+\sqrt{5}}{\sqrt{2(-1+\sqrt{5})}} = \sqrt{\frac{-1+\sqrt{5}}{2}}.$$

$$\sec^2 \alpha = 1 + \tan^2 \alpha = 1 + \frac{-1+\sqrt{5}}{2}$$

$$= \frac{1+\sqrt{5}}{2}.$$

(b) (i) $I_n = \int_0^x \sec^n t \, dt.$

Let $u = \sec^{n-2} t, dv = \sec^2 t \, dt.$

$du = (n-2)\sec^{n-2} t . \tan t . dt, v = \tan t.$

$$I_n = \left[\sec^{n-2} t \tan t\right]_0^x - (n-2)\int_0^x \sec^{n-2} t \tan^2 t \, dt$$

$$= \sec^{n-2} x \tan x - (n-2)\int_0^x \sec^{n-2} t(\sec^2 t - 1)\, dt$$

$$= \sec^{n-2} x \tan x - (n-2)I_n + (n-2)I_{n-2}.$$

$$\therefore (n-1)I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2}.$$

$$\therefore I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}.$$

(ii) $I_4 = \dfrac{\sec^2 \frac{\pi}{3} \tan\frac{\pi}{3}}{3} + \dfrac{2}{3} I_2$

$$= \frac{4\sqrt{3}}{3} + \frac{2}{3} I_2.$$

$$I_2 = \tan\frac{\pi}{3} + 0 = \sqrt{3}.$$

$$\therefore I_4 = \frac{4\sqrt{3}}{3} + \frac{2\sqrt{3}}{3} = 2\sqrt{3}.$$

(c) When $n = 1, x_1 = 2\left(\dfrac{1+\alpha}{1-\alpha}\right)$

$$= 2\left(\frac{1-\frac{1}{3}}{1+\frac{1}{3}}\right)$$

$$= 2\left(\frac{2}{4}\right)$$

$$= 1.$$

True for $n = 1$.

Assume $x_n = 2\left(\dfrac{1+\alpha^n}{1-\alpha^n}\right)$ is true, we are required to prove that it's true for x_{n+1}.

$$x_{n+1} = \frac{4 + x_n}{1 + x_n}$$

$$= \frac{4 + 2\left(\frac{1+\alpha^n}{1-\alpha^n}\right)}{1 + 2\left(\frac{1+\alpha^n}{1-\alpha^n}\right)}$$

$$= \frac{4 - 4\alpha^n + 2 + 2\alpha^n}{1 - \alpha^n + 2 + 2\alpha^n}$$

$$= 2\left(\frac{3 - \alpha^n}{3 + \alpha^n}\right)$$

$$= 2\left(\frac{1 - \alpha^n . \frac{1}{3}}{1 + \alpha^n . \frac{1}{3}}\right)$$

$$= 2\left(\frac{1 + \alpha^{n+1}}{1 - \alpha^{n+1}}\right).$$

$\therefore x_{n+1}$ is true.

$\therefore$ It's true for all $n \geq 1$.

(ii) As $n \to \infty, \alpha^n \to 0, \therefore x_n \to 2.$

Question 8

(a) (i) $t \leq \frac{1}{\sqrt{2}}.$

$$t^2 \leq \frac{1}{2}.$$

$$1 - t^2 \geq 1 - \frac{1}{2} = \frac{1}{2}.$$

$$\frac{1}{1 - t^2} \leq 2.$$

$$\therefore \frac{2t^2}{1 - t^2} \leq 4t^2.$$

Also, since $t \leq \frac{1}{\sqrt{2}}, 1 - t^2 \geq 0, \therefore \frac{2t^2}{1 - t^2} \geq 0.$

(ii) By long division, $\frac{2t^2}{1 - t^2} = -2 + \frac{2}{1 - t^2}$

then by partial fractions, $= -2 + \frac{1}{1 + t} + \frac{1}{1 - t}.$

$$\therefore 0 \leq -2 + \frac{1}{1 + t} + \frac{1}{1 - t} \leq 4t^2.$$

(iii) $0 \leq \int_0^x \left(-2 + \frac{1}{1 + t} + \frac{1}{1 - t}\right) dt \leq \int_0^x 4t^2 \, dt, \; 0 \leq x \leq \frac{1}{\sqrt{2}}.$

$$\left[-2t + \ln(1 + t) - \ln(1 - t)\right]_0^x \leq \left[\frac{4t^3}{3}\right]_0^x.$$

$$-2x + \ln\frac{1 + x}{1 - x} \leq \frac{4x^3}{3}.$$

(iv) $\ln e^{-2x} + \ln\frac{1 + x}{1 - x} \leq \ln e^{\frac{4x^3}{3}}.$

$$\ln\left(\frac{1 + x}{1 - x}\right)e^{-2x} \leq \ln e^{\frac{4x^3}{3}}.$$

$$\therefore \left(\frac{1 + x}{1 - x}\right)e^{-2x} \leq e^{\frac{4x^3}{3}}.$$

(b) (i) $f'(x) = nx^{n-1}e^{-x} - x^n e^{-x}$

$$= x^{n-1}(n - x)e^{-x}.$$

$$f''(x) = n(n-1)x^{n-2}e^{-x} - nx^{n-1}e^{-x} - x^{n-1}(n - x)e^{-x}$$

$$= x^{n-2}e^{-x}\left(n(n-1) - nx - x(n - x)\right)$$

$$= x^{n-2}e^{-x}\left(x^2 - 2nx + n(n-1)\right).$$

$f''(x) = 0$ gives $x = n \pm \sqrt{n^2 - n(n-1)}$

$$= n \pm \sqrt{n}.$$

$f''(x)$ changes signs at these points so they are points of inflexion.

$\therefore b = n + \sqrt{n}, a = n - \sqrt{n}$ (so that $b > a$).

(ii) $\frac{f(b)}{f(a)} = \frac{\left(n + \sqrt{n}\right)^n e^{-(n+\sqrt{n})}}{\left(n - \sqrt{n}\right)^n e^{-(n-\sqrt{n})}}$

$$= \left(\frac{n + \sqrt{n}}{n - \sqrt{n}}\right)^n e^{-2\sqrt{n}}$$

$$= \left(\frac{1 + \frac{\sqrt{n}}{n}}{1 - \frac{\sqrt{n}}{n}}\right)^n e^{-2\sqrt{n}} = \left(\frac{1 + \frac{1}{\sqrt{n}}}{1 - \frac{1}{\sqrt{n}}}\right)^n e^{-2\sqrt{n}}.$$

(iii) Replacing x by $\frac{1}{\sqrt{n}}$

$\left(\frac{1 + x}{1 - x}\right)e^{-2x} \leq e^{\frac{4x^3}{3}}$ becomes $\left(\frac{1 + \frac{1}{\sqrt{n}}}{1 - \frac{1}{\sqrt{n}}}\right)e^{\frac{-2}{\sqrt{n}}} \leq e^{\frac{4}{3n\sqrt{n}}}.$

$$\therefore \left(\frac{1 + \frac{1}{\sqrt{n}}}{1 - \frac{1}{\sqrt{n}}}\right)^n e^{-2\sqrt{n}} = \left(\left(\frac{1 + \frac{1}{\sqrt{n}}}{1 - \frac{1}{\sqrt{n}}}\right)e^{\frac{-2}{\sqrt{n}}}\right)^n \leq \left(e^{\frac{4}{3n\sqrt{n}}}\right)^n$$

$$= e^{\frac{4n}{3n\sqrt{n}}}$$

$$= e^{\frac{4}{3\sqrt{n}}}.$$

$$\therefore \frac{f(b)}{f(a)} \leq e^{\frac{4}{3\sqrt{n}}}.$$

$\frac{f(b)}{f(a)} \geq 1$ because the numerator is always more than the denominator.

$$\therefore 1 \leq \frac{f(b)}{f(a)} \leq e^{\frac{4}{3\sqrt{n}}}.$$

(iv) As $n \to \infty, e^{\frac{4}{3\sqrt{n}}} \to 1, \therefore \frac{f(b)}{f(a)} = 1.$