Practice Questions

Solutions

Complex Numbers

(a) (i) Let $z = x + iy, |z|^2 = \left(\sqrt{x^2 + y^2}\right)^2 = x^2 + y^2$ while $z\bar{z} = (x + iy)(x - iy) = x^2 + y^2$.

$\therefore |z|^2 = z\bar{z}$

and $\left(\overline{z_1 + z_2}\right) = \overline{(x_1 + iy_1 + x_2 + iy_2)} = \overline{\left(x_1 + x_2 + i\left(y_1 + y_2\right)\right)} = x_1 + x_2 - i\left(y_1 + y_2\right),$

$\bar{z_1} + \bar{z_2} = x_1 - iy_1 + x_2 + iy_2 = x_1 + x_2 - i\left(y_1 + y_2\right)$

$\therefore \left(\overline{z_1 + z_2}\right) = \bar{z_1} + \bar{z_2}.$

(ii) $|z - z_1|^2 + |z - z_2|^2 = (z - z_1)\overline{(z - z_1)} + (z - z_2)\overline{(z - z_2)}$

$= (z - z_1)\left(\bar{z} - \bar{z_1}\right) + (z - z_2)\left(\bar{z} - \bar{z_2}\right)$

$= z\bar{z} + z_1\bar{z_1} - z\bar{z_1} - z_1\bar{z} + z\bar{z} + z_2\bar{z_2} - z\bar{z_2} - z_2\bar{z}$

$= 2z\bar{z} + z_1\bar{z_1} + z_2\bar{z_2} - z\left(\bar{z_1} + \bar{z_2}\right) - \bar{z}\left(z_1 + z_2\right)$

$= 2|z|^2 + |z_1|^2 + |z_2|^2 - z\overline{\left(z_1 + z_2\right)} - \bar{z}\left(z_1 + z_2\right)$

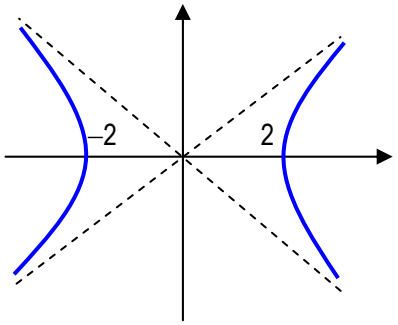
$= 2 + 5^2 + 5^2,$ since $z_1 + z_2 = 0,$ so $\overline{z_1 + z_2} = 0$

$= 52.$

(b) (i) Let $z = x + iy, \mathrm{Re}(z^2) = \mathrm{Re}\left(x^2 - y^2 + 2xyi\right) = x^2 - y^2$.

$\therefore$ The locus of z is the rectangular hyperbola $x^2 - y^2 = 4$.

(ii) $-\pi < \arg z < -\frac{3\pi}{4}, -\frac{\pi}{4} < \arg z < \frac{\pi}{4}, \frac{3\pi}{4} < \arg z \le \pi$.

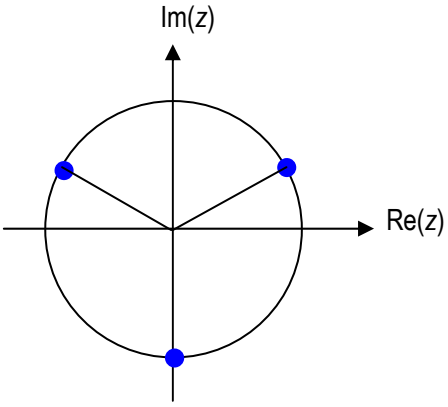


(c) $i = \text{cis}\dfrac{\pi}{2} = \text{cis}\left(\dfrac{\pi}{2} + 2k\pi\right)$.

$\therefore i^{\frac{1}{3}} = \text{cis}\left(\dfrac{\pi}{6} + \dfrac{2k\pi}{3}\right)$, where $k = 0, 1, -1$.

$= \text{cis}\dfrac{\pi}{6}, \text{cis}\dfrac{5\pi}{6}, \text{cis}\left(-\dfrac{\pi}{2}\right)$

$= \dfrac{\sqrt{3}+i}{2}, \dfrac{-\sqrt{3}+i}{2}, -i.$



(d) (i) $u + v = (1-i) + \sqrt{2}(1+i) = \left(1+\sqrt{2}\right) + i\left(\sqrt{2}-1\right)$.

$iv = i\sqrt{2}(1+i) = \sqrt{2}(-1+i)$.

(ii) From the diagram, $ABCD$ is a trapezium, where

$\overrightarrow{AB} = (u+v) - u = v = \sqrt{2}(1+i), \therefore AB = 2$

$\overrightarrow{CB} = (u+v) - v = u = 1-i, \therefore BC = \sqrt{2}$

$\overrightarrow{AD} = u - iv = (1-i) - \sqrt{2}(-1+i) = \left(1+\sqrt{2}\right)(1-i), \therefore AD = \sqrt{2}\left(1+\sqrt{2}\right)$.

$\therefore$ Area of $ABCD = \dfrac{1}{2}(BC + AD)\,AB$

$= \dfrac{1}{2}\left(\sqrt{2} + \sqrt{2}\left(1+\sqrt{2}\right)\right)2 = 2\left(\sqrt{2}+1\right)$.