Extension 1 Assignment.

1) Yr 11 topics

a) Solve $\frac{2x}{x-4} \geq 1$

$2x(x-4) \geq (x-4)^2$

$(x-4)(2x-x+4) \geq 0$

$(x-4)(x+4) \geq 0$

$\therefore x \leq -4$ or $x > 4$ (note: $x \neq 4$)

b) Given P(–3,5) and R(2,–1). Find the coordinates of a point S which divides PR externally in the ratio 2:1.

$x = \frac{-1(-3)+2(2)}{-1+2} = 7, y = \frac{-1(5)+2(-1)}{-1+2} = -7$

$\therefore S(7,-7)$

c) Find the exact value of tan105°.

$\tan(45+60)° = \frac{\tan 45° + \tan 60°}{1 - \tan 45° \tan 60°} = \frac{1+\sqrt{3}}{1-\sqrt{3}} \left(= -2-\sqrt{3}\right)$

d) Show that $\frac{1+\cos 2\theta}{\sin 2\theta} = \cot\theta$.

$\text{LHS} = \frac{1+(2\cos^2\theta - 1)}{2\sin\theta\cos\theta} = \frac{\cos\theta}{\sin\theta} = \cot\theta = \text{RHS}$

e) Find the general solution for $\tan x = \sqrt{3}$.

$x = \frac{\pi}{3} + k\pi, k \in J$.

2) Log, Exp and Trig functions

a) i) Find $\frac{d}{dx}(x\sin x + \cos x)$.

$\frac{d}{dx} = \sin x + x\cos x - \sin x = x\cos x$

ii) Hence, evaluate $\int_{\pi/2}^{\pi} x\cos x \, dx$.

$= \left[x\sin x + \cos x \right]_{\frac{\pi}{2}}^{\pi} = -1 - \frac{\pi}{2}$

b) i) Sketch on the same axes the curves $y = \cos 2x$ and $y = 2\sin^2 x$, for $0 \leq x \leq \pi$.

ii) Find the x-coordinates of the points of intersection, for $0 \leq x \leq \pi$.

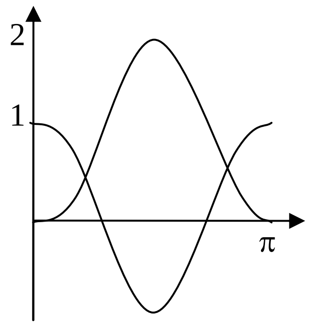
$\cos 2x = 2\sin^2 x$

$1 - 2\sin^2 x = 2\sin^2 x$

$4\sin^2 x = 1$

$\sin x = \pm\frac{1}{2}$

For $0 \leq x \leq \pi, x = \frac{\pi}{6}, \frac{5\pi}{6}$.



c) The area enclosed by the curve $y = \cos\frac{x}{2}$, the x-axis between $x = 0$ and $x = 2\pi$ is rotated about the x-axis. Find the exact volume of the solid produced.

$V = \pi\int_0^{2\pi} \cos^2\frac{x}{2} \, dx = \frac{\pi}{2}\int_0^{2\pi} (1+\cos x) \, dx$

$= \frac{\pi}{2}\left[x + \sin x \right]_0^{2\pi} = \frac{\pi}{2} \times 2\pi = \pi^2$.

d) Write $\sqrt{3}\cos x - \sin x$ in the form $R\cos(x+\alpha)$, hence, solve $\sqrt{3}\cos x - \sin x = 1$ for $0 \leq x \leq 2\pi$.

$\sqrt{3}\cos x - \sin x = 2\cos\left(x + \frac{\pi}{6}\right) = 1$.

$\cos\left(x + \frac{\pi}{6}\right) = \frac{1}{2}$.

$x + \frac{\pi}{6} = \pm\frac{\pi}{3} + k2\pi$.

$x = \pm\frac{\pi}{3} - \frac{\pi}{6} + k2\pi$.

For $0 \leq x \leq 2\pi, x = \frac{\pi}{6}, \frac{3\pi}{2}$.

3) Inverse functions

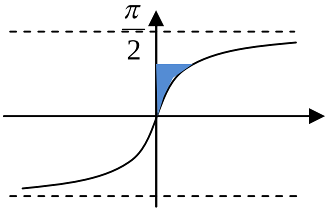
a) i) Sketch the curve $y = \tan^{-1} x$.

ii) Show that the area obtained by this curve, the y-axis and the line $y = \frac{\pi}{4}$ is $\ln\sqrt{2}$.

$A = \int_0^{\frac{\pi}{4}} x dy = \int_0^{\frac{\pi}{4}} \tan y \, dy$

$= \left[\ln(\sec y) \right]_0^{\frac{\pi}{4}}$

$= \ln\sqrt{2}$



b) i) Explain why $y = 4x - x^2$ does not have an inverse function.

Because it fails the horizontal line test.

ii) Find the largest domain so that the restricted function may have an inverse function.

Either $x \geq 2$ or $x \leq 2$.

iii) Find the inverse function.

$x = 4y - y^2$

$y^2 - 4y + x = 0$.

$y = 2 \pm \sqrt{4-x}$.

Take the + sign if $x \geq 2$ is chosen, or – sign if $x \leq 2$ is chosen.

iv) Find the point of intersection of the inverse function and the restricted original function.

$4x - x^2 = x$

$x^2 - 3x = 0$.

$\therefore x = 0$ or 3.

$\therefore x = 3$ if $x \geq 2$ is chosen or $x = 0$ if $x \leq 2$ is chosen.

v) α is a point that does not belong to the domain in (ii), find $f^{-1}\left(f(\alpha)\right)$

Let $f(f^{-1}(\alpha)) = \beta$.

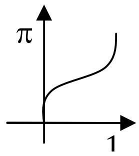
Since $\dfrac{\alpha+\beta}{2} = 2, \beta = 4-\alpha$.

c) i) State the domain and range.

D: $0 \le x \le 1$

R: $0 \le y \le \pi$

ii) Sketch the graph of $f(x) = 2\sin^{-1}\sqrt{x}$.



d) Find the exact values of

i) $\cos^{-1}\left(\dfrac{-\sqrt{3}}{2}\right) = \dfrac{5\pi}{6}$

ii) $\sin\left(\tan^{-1}\dfrac{9}{40}\right) = \dfrac{9}{41}$

iii) $\cos\left(2\cos^{-1}\dfrac{3}{4}\right) = 2\times\dfrac{9}{16} - 1 = \dfrac{1}{8}$

e) Differentiate

i) $\dfrac{d}{dx}\cos^{-1}2x = \dfrac{-2}{\sqrt{1-4x^2}}$

ii) $\dfrac{d}{dx}\sin^{-1}\dfrac{x}{2} = \dfrac{1}{\sqrt{4-x^2}}$

iii) $\dfrac{d}{dx}\cos^{-1}e^x = \dfrac{-e^x}{\sqrt{1-e^{2x}}}$

iv) $\dfrac{d}{dx}\ln(\tan^{-1}x) = \dfrac{1}{(1+x^2)\tan^{-1}x}$.

f) Find

i) $\displaystyle\int \frac{1}{\sqrt{9-x^2}}dx = \sin^{-1}\frac{x}{3} + C$

ii) $\displaystyle\int \frac{x}{\sqrt{1-36x^2}}dx = -\frac{1}{36}\sqrt{1-36x^2} + C$.

iii) $\displaystyle\int_{-2}^{2} \frac{dx}{x^2+4} = 2\times\frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_0^2 = \frac{\pi}{4}$.

iv) $\displaystyle\int \frac{dx}{\sqrt{16-9x^2}} = \frac{1}{3}\sin^{-1}\frac{3x}{4} + C$.

g) Find the exact values of

i) $\tan^{-1}\left(\tan\dfrac{2\pi}{3}\right) = -\dfrac{\pi}{3}$.

ii) $\cos^{-1}\left(\cos\dfrac{5\pi}{4}\right) = \dfrac{3\pi}{4}$.

iii) $\tan\left(\sin^{-1}\dfrac{5}{13}\right) = \dfrac{5}{12}$.

iv) $\sin^{-1}\dfrac{1}{2} + \cos^{-1}\dfrac{1}{3} > \dfrac{\pi}{2}$ so $\cos^{-1}$ MUST be used.

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \dfrac{\sqrt{3}}{2}\dfrac{1}{3} - \dfrac{1}{2}\dfrac{\sqrt{8}}{3} = \dfrac{\sqrt{3}-2\sqrt{2}}{6}$.

$\therefore \alpha+\beta = \cos^{-1}\dfrac{\sqrt{3}-2\sqrt{2}}{6}$.

4) Integration

Evaluate the following integrals.

i) $\displaystyle\int \frac{x}{\sqrt{1-x}}dx$ using the substitution $x = 1-u^2$.

$x = 1-u^2, dx = -2u\,du$.

$$\int \frac{x}{\sqrt{1-x}}dx = \int \frac{1-u^2}{u}(-2udu)$$

$$= -2\int\left(1-u^2\right)du$$

$$= -2\left(u - \frac{u^3}{3}\right)$$

$$= \frac{2}{3}\sqrt{(1-x)^3} - 2\sqrt{1-x} + C.$$

ii) $\displaystyle\int x^3\sqrt{x^2-4}dx$ using the substitution $u = x^2-4$.

$u = x^2-4, du = 2x\,dx$.

$$\int x^2\sqrt{x^2-4}\times x\,dx = \int (u+4)\sqrt{u}\times\frac{1}{2}du$$

$$= \frac{1}{2}\int\left(u^{\frac{3}{2}} + 4u^{\frac{1}{2}}\right)du$$

$$= \frac{1}{2}\left(\frac{2u^{\frac{5}{2}}}{5} + \frac{8u^{\frac{3}{2}}}{3}\right) + C$$

$$= \frac{\sqrt{(x^2-4)^5}}{5} + \frac{4\sqrt{(x^2-4)^3}}{3} + C.$$

iii) $\displaystyle\int_{\frac{1}{2}}^{1} \frac{dx}{x^2\sqrt{1-x^2}}$ using the substitution $x = \sin\theta$.

$x = \sin\theta, dx = \cos\theta\,d\theta$.

When $x = \dfrac{1}{2}, \theta = \dfrac{\pi}{6}$. When $x = 1, \theta = \dfrac{\pi}{2}$.

$$\int_{\frac{1}{2}}^{1} \frac{dx}{x^2\sqrt{1-x^2}} = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos\theta}{\sin^2\theta\cos\theta}d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^2\theta d\theta$$

$$= \left[-\cot\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= 0 - (-\sqrt{3}) = \sqrt{3}.$$

iv) $\displaystyle\int_7^8 \frac{dx}{\sqrt{x^2-6x-7}}$ using the substitution $u = x-3$.

$u = x-3, du = dx, x^2-6x-7 = u^2-16$.

When $x = 7, u = 4$. When $x = 8, u = 5$.

$$\int_7^8 \frac{dx}{\sqrt{x^2-6x-7}} = \int_4^5 \frac{du}{\sqrt{u^2-16}}$$

$$= \left[\ln\left(u + \sqrt{u^2-16}\right)\right]_4^5$$

$$= \ln\frac{5+3}{4}$$

$$= \ln 2.$$

5) Binomial, Permutation, Combination and Probability

a) A particular term of the expansion $\left(2x^2+\frac{A}{x}\right)^{10}$ is $\frac{15}{2x}$, evaluate *A*.

In ${}^{10}C_r(2x^2)^{10-r}\left(\frac{A}{x}\right)^r$, $x^{20-3r}=x^{-1}$.

$20-3r=-1, \therefore r=7$.

$\therefore {}^{10}C_7 2^3 A^7=\frac{15}{2}$.

$A^7=\frac{15}{{}^{10}C_7 2^4}=\frac{1}{128}$.

$\therefore A=\frac{1}{\sqrt[7]{128}}=\frac{1}{2}$.

b) A bag contains 2 white and 3 red balls. Two balls are drawn at random from the bag. Find the probability that both balls are red. Consider both cases with or without replacement.

With replacement, Pr(both red) = $\left(\frac{3}{5}\right)^2=\frac{9}{25}$.

Without replacement, Pr(both red) = $\frac{{}^3C_2}{{}^5C_2}=\frac{3}{10}$.

c) i) What is the probability that if the letters of the word ENTERTAIN are arranged in a row randomly, then the letters T's will not be together?

$1-\frac{\frac{8!}{2!2!}}{\frac{9!}{2!2!2!}}=1-\frac{2}{9}=\frac{7}{9}$ OR by Insertion $\frac{\frac{7!}{2!2!}\times {}^8C_2}{\frac{9!}{2!2!2!}}=\frac{7}{9}$.

ii) Repeat the question for ENTERTAINMENT.

By Insertion method, $\frac{\frac{10!}{3!3!}\times {}^{11}C_3}{\frac{13!}{3!3!3!}}=\frac{15}{26}$.

d) It was determined over a long period of time that a particular machine produces three defective items in every one hundred. If fifty items made by this machine are chosen at random find the probability that at most two items are defective.

$(0.97)^{50}+{}^{50}C_1(0.03)(0.97)^{49}+{}^{50}C_2(0.03)^2(0.97)^{48}$
$=0.8108$.

e) i) Find the coefficient of x^6 in the expansion of $(2x-3)^9$.

${}^9C_6 2^6(-3)^3=-145152$

ii) Find the term independent of *x* in the expansion of $\left(x^2+\frac{1}{2x^2}\right)^{12}$

In ${}^{12}C_r(x^2)^{12-r}\left(\frac{1}{2x^2}\right)^r$, $x^{24-4r}=x^0$.

$24=4r$

$r=6$.

$\therefore$ The term independent of *x* is $\frac{{}^{12}C_6}{2^6}=\frac{231}{16}$.

iii) Find the greatest coefficient in the expansion of $(2+3x)^{12}$. Hence, algebraically show that it can be written as $\left(\binom{6}{6}\binom{6}{1}+\binom{6}{5}\binom{6}{2}+\binom{6}{4}\binom{6}{3}\right)2^6 3^7$.

$$\frac{{}^{12}C_r 2^{12-r}3^r}{{}^{12}C_{r-1}2^{13-r}3^{r-1}}=\frac{\frac{12!}{r!(12-r)!}}{\frac{12!}{(r-1)!(13-r)!}}\frac{3}{2}=\frac{3(13-r)}{2r}\ge 1$$

$39-3r\ge 2r$

$39\ge 5r$.

$r\le\frac{39}{5}=7.8, \therefore r=7$.

$\therefore$ The greatest coefficient is ${}^{12}C_7 2^5 3^7$.

The coefficient of x^7 in $(2+3x)^6(2+3x)^6$ is

$\binom{6}{6}3^6\binom{6}{1}2^5 3^1+\binom{6}{5}2^1 3^5\binom{6}{2}2^4 3^2+\binom{6}{4}2^2 3^4\binom{6}{3}2^3 3^3$

$+\ldots+\binom{6}{1}2^5 3^1\binom{6}{6}3^6$, which is

$\left(\binom{6}{6}\binom{6}{1}+\binom{6}{5}\binom{6}{2}+\binom{6}{4}\binom{6}{3}+\binom{6}{3}\binom{6}{4}+\binom{6}{2}\binom{6}{5}+\binom{6}{1}\binom{6}{6}\right)2^5 3^7$,

i.e. $\left(\binom{6}{6}\binom{6}{1}+\binom{6}{5}\binom{6}{2}+\binom{6}{4}\binom{6}{3}\right)2^6 3^7$.

6) Applications of Calculus

a) Water is pouring steadily at the rate of 1 m^3/min into an inverted conical reservoir whose semi-vertical angle is 60°.

i) Show that $r=\sqrt{3}h$.

$\tan 60°=\frac{r}{h}, \therefore r=h\tan 60°=\sqrt{3}h.$

ii) When the water is 2 m deep, find the rate at which the water level is rising.

$V=\frac{1}{3}\pi r^2 h=\pi h^3, \therefore \frac{dV}{dh}=3\pi h^2$.

$\frac{dh}{dt}=\frac{dh}{dV}\frac{dV}{dt}=\frac{1}{3\pi h^2}\times 1=\frac{1}{3\pi h^2}$.

When $h=2, \frac{dh}{dt}=\frac{1}{12\pi}$ m/min.

b) A man throws a ball from the height of 2 m to the roof of a 15 metre high building. He throws the ball at an initial velocity of 25m/s, and he is 20 m from the base of the building. Between which two angles of projection must he throw the ball to ensure that it lands on the roof of the building?

You are given these equations of motion:

$x=25t\cos\theta, y=25t\sin\theta-5t^2+2$.

When $x=20$, $y\ge 15$,

$20=25t\cos\theta, \therefore t=\frac{20}{25\cos\theta}=\frac{4}{5\cos\theta}$.

$$25\times\frac{4}{5\cos\theta}\sin\theta-5\frac{16}{25\cos^2\theta}+2\geq 15$$

$$20\tan\theta-\frac{16}{5}\sec^2\theta-13\geq 0.$$

$$100\tan\theta-16(\tan^2\theta+1)-65\geq 0.$$

$$16\tan^2\theta-100\tan\theta+81\geq 0.$$

$$\Delta=100^2-4(16)(81)=4816.$$

For $0<\theta<90°, \frac{100-\sqrt{4816}}{32}\leq\tan\theta\leq\frac{100+\sqrt{4816}}{32}$.

$44°\leq\theta\leq 79°$.

c) The velocity v m/s of a particle moving in SHM along the x-axis is given by $v^2=-28+11x-x^2$, where x is in metres.

i) Between which two points is the particle oscillating?

$v^2=-28+11x-x^2=(7-x)(-4+x)$.

$v^2\geq 0, \therefore 4\leq x\leq 7$.

ii) Find the centre of motion.

Centre is at $x=\frac{4+7}{2}=5.5$ m.

iii) Find the maximum speed of the particle.

When $x=5.5, v^2=2.25, \therefore v_{max}=1.5$ m/s.

iv) Find the acceleration of the particle in terms of x.

$$a=\frac{d}{dx}\left(\frac{1}{2}v^2\right)=\frac{11}{2}-x.$$

d) A particle is moving in a straight line. The particle starts from the origin with speed of 4 cm/s and you are given that its acceleration is $a=2(x-2)^3$ cm^2/s.

i) Show that the velocity of the particle is $v=(x-2)^2$.

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right)=2(x-2)^3.$$

$$\frac{1}{2}v^2=\frac{(x-2)^4}{2}+C.$$

When $x=0, v=4, \therefore C=0$.

$\therefore v^2=(x-2)^4, \therefore v=(x-2)^2$.

ii) Find an expression for the displacement in terms of time.

$$\frac{dx}{dt}=(x-2)^2.$$

$$\frac{dt}{dx}=\frac{1}{(x-2)^2}.$$

$$t+C=\frac{-1}{x-2}.$$

When $t=0, x=0, \therefore C=\frac{1}{2}$.

$$t+\frac{1}{2}=-\frac{1}{x-2}.$$

$$x-2=-\frac{1}{t+\frac{1}{2}}=-\frac{2}{2t+1}.$$

$$\therefore x=2-\frac{2}{2t+1}.$$

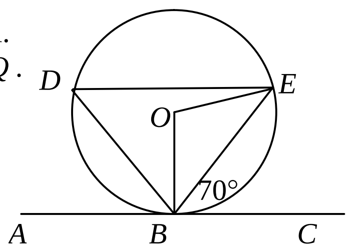
iii) Briefly describe the motion of the particle.

As $t\to\infty, \frac{2}{2t+1}\to 0, \therefore x\to 2$: the particle travels from the origin to its limiting position of x = 2 m.

7) Circle Geometry

a) AC is a tangent. O is the centre. $\angle CBE$ = 70°.
If DE is parallel to AC, find $\angle DEO$.
$\angle BDE$ = 70° (alternate angles on parallel lines)
$\angle OBE$ = 90° (radius is perpendicular to tangent)
$\angle OBE$ = 20° (adjacent angles)
$\angle OEB$ = 20° (base angle in isosceles Δ, $OE = OB$)
$\therefore\angle DEO$ = 70° – 20° = 50°.

b) $PQRS$ is a cyclic quadrilateral. $PQ = PS$, $\angle PSQ = \theta$.
Prove that $\angle QRS = 2\theta$.
$\angle QPS = 180° - 2\theta$ (angle sum in isosceles Δ).
$\therefore\angle QRS = 2\theta$ (opposite angles in cyclic quadrilateral are supplementary)



c) $PQRS$ is a cyclic quadrilateral.
Prove that $PR\sin P = QS\sin Q$.

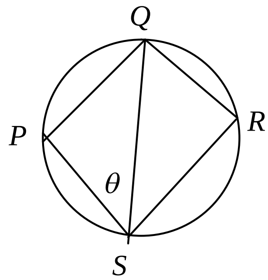
In $\Delta PQR, \frac{\sin Q}{PR}=\frac{\sin QRP}{PQ}$.

In $\Delta PQS, \frac{\sin P}{QS}=\frac{\sin PSQ}{PQ}$.

But $\angle PSQ=\angle QRP$ (angles subtending same arc are equal)

$\therefore\frac{\sin Q}{PR}=\frac{\sin P}{QS}$.

$\therefore PR\sin P=QS\sin Q$



8) Parametric equations

$P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are points on the parabola $4ay=x^2$ such that $PS+SQ=4a$.
Prove that the locus of the midpoint of PQ is a segment of a line.

$PS=PM=ap^2-(-a)=ap^2+a$.

$\therefore ap^2+a+aq^2+a=4a$ (since $PS+QS=4a$)

$\therefore p^2+q^2=2$.

The midpoint of PQ has coordinates

$$x=\frac{2ap+2aq}{2}=a(p+q)$$

$$y=\frac{ap^2+aq^2}{2}=\frac{a(p^2+q^2)}{2}=\frac{2a}{2}=a$$

$\therefore$The locus is the line $y = a$, but as the midpoint cannot lie outside the parabola, its locus is restricted by $y = a$, $-2a\leq x\leq 2a$.