

Chapter 1

15 marks

(a) The curve $y = x^3 + x^2 - 5x + 1$ has two turning points A and B.

(i) What are the coordinates of A and B?

$y' = 3x^2 + 2x - 5 = (3x + 5)(x - 1) = 0$ gives $x = 1, -\frac{5}{3} \therefore A(1, -2), B(-\frac{5}{3}, \frac{202}{27}) \approx (1.7, 7.5)$ 1m

(ii) For what values of k has the equation $x^3 + x^2 - x + 2 = k$ three real solutions?

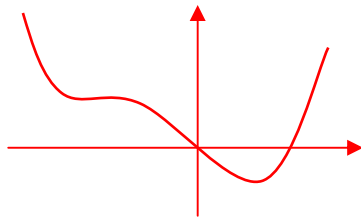
$y' = 3x^2 + 2x - 1 = (3x - 1)(x + 1) = 0$ gives $x = \frac{1}{3}, -1, \therefore$ TPs $(\frac{1}{3}, \frac{49}{27}), (-1, 3), \therefore \frac{49}{27} < k < 3$ 2m

(b) Consider the curve $y = x^4 + 4x^3 - 16x$.

(i) Find and factorise y' completely and hence determine the location and nature of any stationary points.

$y' = 4x^3 + 12x^2 - 16 = 4(x^3 + 3x^2 - 4) = 4(x - 1)(x^2 + 4x + 4) = 4(x - 1)(x + 2)^2$
For determining the nature use the graph of y' or the table values of y' (not y'' , because $y'' = 0$ too, when $x = -2$), $x = 1$ is min 2m
and $x = -2$ is hpoi.

(ii) Sketch the curve.



2m

(c) Let $f(x) = \frac{2x}{x^2 - 1}$.

(i) For what values of x is f(x) undefined? Also state the equation of the horizontal asymptote.

$x = \pm 1$. Horizontal asymptote $y = 0$ 1m

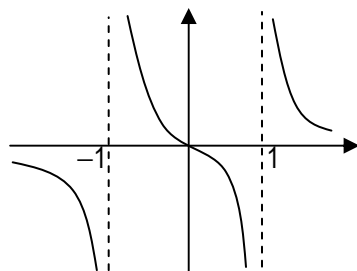
(ii) Show that $y = f(x)$ is an odd function.

$f(-x) = \frac{2(-x)}{(-x)^2 - 1} = -\frac{2x}{x^2 - 1} = -f(x)$ 1m

(iii) Show that $f'(x) < 0$ at all values of x for which the function is defined.

$f'(x) = \frac{2(x^2 - 1) - 4x^2}{(x^2 - 1)^2} = \frac{-2x^2 - 2}{(x^2 - 1)^2} = \frac{-2(x^2 + 1)}{(x^2 - 1)^2} < 0$ since $x^2 + 1 > 0$ 1m

(iv) Hence, sketch $y = f(x)$.



1m

(d) Let $f(x) = 3x^5 - 10x^3 + 45x$.

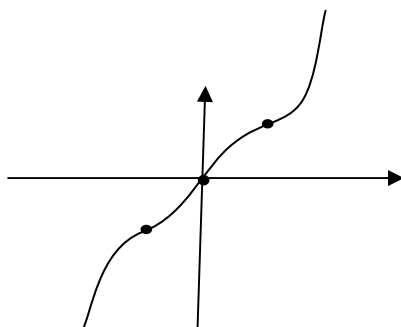
(i) Show that $f'(x) > 0$ for all x.

$f'(x) = 15x^4 - 30x^2 + 45 = 15(x^4 - 2x^2 + 3) = 15(x^2 - 1)^2 + 30 > 0$ since $(x^2 - 1)^2 \geq 0$ 1m

(ii) For what values of x is $f''(x)$ positive?

$f''(x) = 60x^3 - 60x = 60x(x^2 - 1), f''(x) > 0$ when $x > 1$ or $-1 < x < 0$ 1m

(iii) Sketch the graph of $y = f(x)$.



2m

Chapter 2

25 marks

(a) (i) Find the area of the region bounded by $y = x^2 - 4x$, $2 \leq x \leq 5$ and the x -axis.

$$\left| \int_2^4 (x^2 - 4x) dx \right| + \int_4^5 (x^2 - 4x) dx = \left[\frac{x^3}{3} - 2x^2 \right]_2^4 + \left[\frac{x^3}{3} - 2x^2 \right]_4^5 = \left(\frac{8}{3} - 8 \right) - \left(\frac{64}{3} - 32 \right) + \left(\frac{125}{3} - 50 \right) - \left(\frac{64}{3} - 32 \right) = \frac{23}{3} \quad 3m$$

(ii) Find the volumes of the solids generated when the region enclosed by the curve

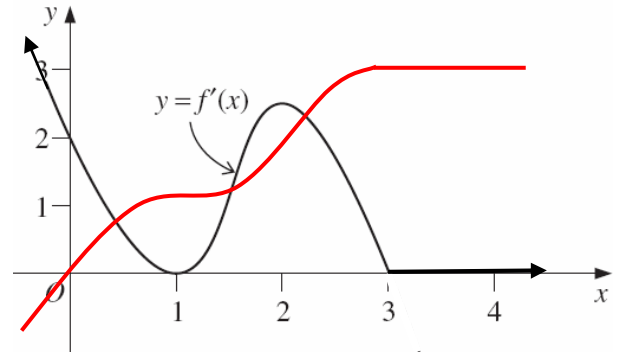
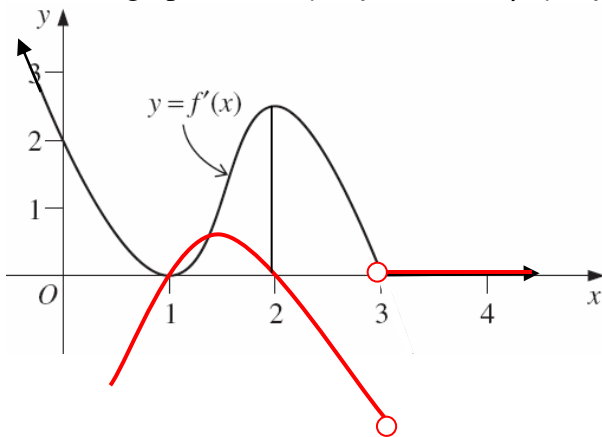
$16y = x^2 + 16$, $-4 \leq x \leq 4$ and the line $y = 2$, is rotated about (α) the x -axis, (β) the y -axis.

$$\begin{aligned} \alpha) V &= \pi \int_{-4}^4 2^2 dx - \pi \int_{-4}^4 \frac{(x^2 + 16)^2}{16^2} dx = 2\pi \int_0^4 4 dx - \frac{\pi}{128} \int_0^4 (x^4 + 32x^2 + 256) dx = 2\pi [4x]_0^4 - \frac{\pi}{128} \left[\frac{x^5}{5} + \frac{32x^3}{3} + 256x \right]_0^4 \\ &= 32\pi - \frac{224\pi}{15} = \frac{256\pi}{15} \quad 4m \end{aligned}$$

$$\beta) V = \pi \int_1^2 (16y - 16) dy = \pi [8y^2 - 16y]_1^2 = \pi((0) - (8 - 16)) = 8\pi \quad 2m$$

(b) The diagram below shows a sketch of the gradient function $f'(x)$ of the curve $y = f(x)$.

Sketch the graphs of (α) $y = f''(x)$ and (β) $y = f(x)$ given that $f(0) = 0$.



4m

(c) Evaluate the following integrals using the substitution given.

i) $\int \frac{x}{\sqrt{1+x^2}} dx = \sqrt{1+x^2} + C \quad 1m$

ii) $\int x^3 \sqrt{x^2 - 4} dx$ Let $u = x^2 - 4, du = 2x dx \therefore x^2 = u + 4$.

$$\therefore \frac{1}{2} \int (u+4)\sqrt{u} du = \frac{1}{2} \int (u^{\frac{3}{2}} + 4u^{\frac{1}{2}}) dx = \frac{1}{2} \left(\frac{2u^{\frac{5}{2}}}{5} + \frac{8u^{\frac{3}{2}}}{3} \right) = \frac{\sqrt{u^5}}{5} + \frac{4\sqrt{u^3}}{3} = \frac{\sqrt{(x^2-4)^5}}{5} + \frac{4\sqrt{(x^2-4)^3}}{3} + C. \quad 3m$$

iii) $\int_0^2 \frac{2x^3}{(x^2+1)^3} dx$ $u = x^2 + 1, \therefore x^2 = u - 1, du = 2x dx$. When $x = 0, u = 1$, when $x = 2, u = 5$

$$\int_1^5 \frac{u-1}{u^3} du = \int_1^5 \left(\frac{1}{u^2} - \frac{1}{u^3} \right) du = \left[-\frac{1}{u} + \frac{1}{2u^2} \right]_1^5 = \left(-\frac{1}{5} + \frac{1}{50} \right) - \left(-1 + \frac{1}{2} \right) = \frac{8}{25} \quad 3m$$

iv) $\int_{-1}^0 15x\sqrt{1+x} dx$ $x = u^2 - 1, \therefore dx = 2udu$. When $x = 0, u^2 = 1, \therefore u = 1$, when $x = -1, u = 0$

$$\int_0^1 15(u^2 - 1)u \cdot 2udu = 30 \int_0^1 (u^4 - u^2) du = 30 \left[\frac{u^5}{5} - \frac{u^3}{3} \right]_0^1 = 30 \left(\frac{1}{5} - \frac{1}{3} \right) = -4. \quad 3m$$

(d) Given that $\int_9^{10} \frac{1}{1+x} dx = \log_e 1.1$, explain why $1.1^{10} < e < 1.1^{11}$ by using a graphical means.

When $x = 9, \frac{1}{1+x} = \frac{1}{10}$, when $x = 10, \frac{1}{1+x} = \frac{1}{11} \therefore \frac{1}{11} \times 1 < \log_e 1.1 < \frac{1}{10} \times 1$, since the area is higher than the lower bound and lower than the upper bound rectangles.

$$\therefore e^{\frac{1}{11}} < 1.1 < e^{\frac{1}{10}}. \quad 2m$$

$$e^{\frac{1}{11}} < 1.1 \text{ gives } e < 1.1^{11} \text{ and } e^{\frac{1}{10}} < 1.1 \text{ gives } e < 1.1^{10} \therefore 1.1^{10} < e < 1.1^{11}$$

Chapter 3: Exponential and Logarithmic Functions (20 marks)

1) Solve $2^x = 3$, expressing your answer correct to 2 decimal places. $x = \log_2 3 = \frac{\log 3}{\log 2} = 1.58$ ♪

2) Differentiate with respect to x :

a) $y = \sqrt{e^{3-2x}} = e^{\frac{3-2x}{2}}$, $\therefore y' = -\sqrt{e^{3-2x}}$ ♪

b) $y = \ln(4x-1)$, $y' = \frac{4}{4x-1}$ ♪

c) $y = x \log_e x$, $y' = \ln x + 1$ ♪

d) $y = \log_{10}(5x-2) = \frac{\ln(5x-2)}{\ln 10}$, $y' = \frac{5}{\ln 10(5x-2)}$ ♪

e) $y = \ln \frac{x-2}{x^2-3} = \ln(x-2) - \ln(x^2-3)$, $y' = \frac{1}{x-2} - \frac{2x}{x^2-3}$ ♪

f) $y = e^{\ln(x^5+6)} = x^5 + 6$, $y' = 5x^4$ ♪

g) $y = 3^{x^5+4} = e^{(x^5+4)\ln 3}$, $y' = 5x^4 \times 3^{x^5+4}$ ♪

h) $y = 3x^2 \log_e x$, $y' = 6x \ln x + 3x$ ♪

3) Find

a) $\int x e^{2x^2} dx = \frac{1}{4} e^{2x^2} + C$ ♪

b) $\int \frac{x}{3-x^2} dx = -\frac{1}{2} \ln(3-x^2) + C$ ♪

c) $\int_0^{\ln 2} \frac{e^x}{e^x+1} dx = \left[\ln(e^x+1) \right]_0^{\ln 2} = \ln 3 - \ln 2$ ♪

d) $\int_0^2 \frac{x}{(x^2+1)^3} dx = \frac{1}{2} \left[\frac{(x^2+1)^{-2}}{-2} \right]_0^2 = \frac{1}{4} \left(1 - \frac{1}{25} \right) = \frac{6}{25}$ ♪

4) For the curve $y = x e^{2x}$

i) Find any stationary points $y' = e^{2x} + 2x e^{2x} = e^{2x}(1+2x)$, $y' = 0$ gives $x = -\frac{1}{2}$, \therefore TP $\left(-\frac{1}{2}, -\frac{1}{2e} \right)$ ♪

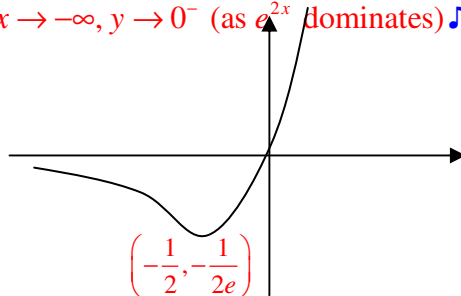
ii) Find any points of inflexion

$y'' = 2e^{2x}(1+2x) + 2e^{2x} = 4e^{2x}(1+x)$, $y'' = 0$ gives $x = -1$, \therefore POI $(-1, -e^{-2})$ ♪

iii) Discuss the behaviour of the curve for positively and negatively large values of x

As $x \rightarrow +\infty$, $y \rightarrow +\infty$; as $x \rightarrow -\infty$, $y \rightarrow 0^-$ (as e^{2x} dominates) ♪

iv) Sketch the curve ♪



5) Find $\frac{d^2}{dx^2}(e^{x^3})$

$\frac{d}{dx}(e^{x^3}) = 3x^2 e^{x^3}$; $\frac{d}{dx}(3x^2 e^{x^3}) = 6x e^{x^3} + 9x^4 e^{x^3}$ ♪

Chapter 4: Trigonometric functions

1. (i) Prove the identity $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

3 marks

$$\begin{aligned}\sin 3\theta &= \sin(\theta + 2\theta) = \sin\theta \cos 2\theta + \cos\theta \sin 2\theta \\ &= \sin\theta(1 - 2\sin^2\theta) + \cos\theta 2\sin\theta \cos\theta \\ &= \sin\theta - 2\sin^3\theta + 2\sin\theta \cos^2\theta \\ &= \sin\theta - 2\sin^3\theta + 2\sin\theta(1 - \sin^2\theta) \\ &= \sin\theta - 2\sin^3\theta + 2\sin\theta - 2\sin^3\theta \\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

- (ii) Hence solve the equation $\sin 3\theta = 2\sin\theta$ for $0 \leq \theta \leq 360^\circ$.

6 marks

$$3\sin\theta - 4\sin^3\theta = 2\sin\theta$$

$$\sin\theta - 4\sin^3\theta = 0$$

$$\sin\theta(1 - 4\sin^2\theta) = 0$$

$$\sin\theta(1 - 2\sin\theta)(1 + 2\sin\theta) = 0$$

$$\sin\theta = 0 \text{ gives } \theta = 0, \pi, 2\pi$$

$$1 - 2\sin\theta = 0 \text{ or } \sin\theta = \frac{1}{2} \text{ gives } \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$1 + 2\sin\theta = 0 \text{ or } \sin\theta = -\frac{1}{2} \text{ gives } \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\therefore \theta = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}, 2\pi.$$

2. Consider the diagram, which shows a vertical tower OT of height h metres, a fixed point A , and a variable point P that is constrained to move so that angle AOP is 60° . The angle of elevation of T from A is 45° . Let the angle of elevation of T from P be α radians and let angle ATP be θ .

- (i) By considering triangle AOP , show that $AP^2 = h^2 + h^2 \cot^2 \alpha - h^2 \cot \alpha$.

2 marks

$$\tan 45^\circ = \frac{OT}{OA}, \therefore OA = OT \cot 45^\circ = OT = h$$

$$\tan \alpha = \frac{OT}{OP}, \therefore OP = OT \cot \alpha = h \cot \alpha$$

$$AP^2 = OT^2 + OP^2 - 2OT \cdot OP \cdot \cos 60^\circ = h^2 + h^2 \cot^2 \alpha - 2h^2 \cot \alpha \cdot \frac{1}{2} = h^2 + h^2 \cot^2 \alpha - h^2 \cot \alpha$$

- (ii) By finding a second expression for AP^2 , deduce that $\cos \theta = \frac{1}{\sqrt{2}} \sin \alpha + \frac{1}{2\sqrt{2}} \cos \alpha$.

6 marks

$$\sin 45^\circ = \frac{OT}{AT}, \therefore AT = \frac{OT}{\sin 45^\circ} = \sqrt{2}h$$

$$\sin \alpha = \frac{OT}{PT}, \therefore PT = \frac{OT}{\sin \alpha} = \frac{h}{\sin \alpha}$$

$$AP^2 = AT^2 + PT^2 - 2AT \cdot PT \cdot \cos \theta = 2h^2 + \frac{h^2}{\sin^2 \alpha} - \frac{2\sqrt{2}h^2}{\sin \alpha} \cos \theta$$

$$\therefore h^2 + h^2 \cot^2 \alpha - h^2 \cot \alpha = 2h^2 + \frac{h^2}{\sin^2 \alpha} - \frac{2\sqrt{2}h^2}{\sin \alpha} \cos \theta \text{ since both } = AP^2$$

$$\therefore \cot^2 \alpha - \cot \alpha = 1 + \operatorname{cosec}^2 \alpha - \frac{2\sqrt{2}}{\sin \alpha} \cos \theta$$

$$\therefore \frac{2\sqrt{2}}{\sin \alpha} \cos \theta = 1 + \operatorname{cosec}^2 \alpha - \cot^2 \alpha + \cot \alpha = 2 + \cot^2 \alpha - \cot^2 \alpha + \cot \alpha = 2 + \cot \alpha$$

$$\therefore \cos \theta = \frac{\sin \alpha}{2\sqrt{2}} (2 + \cot \alpha) = \frac{1}{\sqrt{2}} \sin \alpha + \frac{1}{2\sqrt{2}} \cos \alpha$$

(iii) Write $\frac{1}{\sqrt{2}} \sin \alpha + \frac{1}{2\sqrt{2}} \cos \alpha$ in the form $R \sin(\alpha + \beta)$, hence state the maximum value of $\cos \theta$.

$$R = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{8}} = \sqrt{\frac{5}{8}}, \tan \beta = \frac{\frac{1}{2\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \frac{1}{2}, \therefore \beta = \tan^{-1} \frac{1}{2}$$

4 marks

$$\frac{1}{\sqrt{2}} \sin \alpha + \frac{1}{2\sqrt{2}} \cos \alpha = \sqrt{\frac{5}{8}} \sin\left(\alpha + \tan^{-1} \frac{1}{2}\right)$$

$$\therefore \text{The maximum value of } \frac{1}{\sqrt{2}} \sin \alpha + \frac{1}{2\sqrt{2}} \cos \alpha \text{ is } \sqrt{\frac{5}{8}}$$

3. $\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y$ 3 marks

$$\begin{aligned} \text{LHS} &= (\cos x \cos y - \sin x \sin y)(\cos x \cos y + \sin x \sin y) = \cos^2 x \cos^2 y - \sin^2 x \sin^2 y \\ &= \cos^2 x \cos^2 y - (1 - \cos^2 x)(1 - \cos^2 y) \\ &= \cos^2 x \cos^2 y - 1 + \cos^2 x + \cos^2 y - \cos^2 x \cos^2 y \\ &= -1 + \cos^2 x + \cos^2 y = -1 + \cos^2 x + 1 - \sin^2 y \\ &= \cos^2 x - \sin^2 y = \text{RHS.} \end{aligned}$$

4. Solve the following equations, for $0 \leq x \leq 360^\circ$

(i) $\sin^2 x - 5 \sin x - 2 \cos^2 x = 0$ 4 marks

$$\sin^2 x - 5 \sin x - 2(1 - \sin^2 x) = 0$$

$$3 \sin^2 x - 5 \sin x - 2 = 0$$

$$(3 \sin x + 1)(\sin x - 2) = 0$$

$$\sin x = -\frac{1}{3} \text{ or } \sin x = 2 \text{ (impossible)}$$

$$\therefore x = -19.5^\circ$$

$$\therefore x = 199.5^\circ, 340.5^\circ$$

(ii) $\cot x - 3 \tan x - 2 = 0$ 4 marks

$$\frac{1}{\tan x} - 3 \tan x - 2 = 0$$

$$3 \tan^2 x + 2 \tan x - 1 = 0$$

$$(3 \tan x - 1)(\tan x + 1) = 0$$

$$\tan x = \frac{1}{3} \text{ or } -1$$

$$\text{Solving } \tan x = \frac{1}{3} \text{ gives } x = 18.5^\circ, 198.5^\circ$$

$$\text{Solving } \tan x = -1 \text{ gives } x = 135^\circ, 315^\circ$$

(iii) $\cos^2 x - 3 \sin^2 x - \sin 2x = 0$ 2 marks

$$\cos^2 x - 3 \sin^2 x - 2 \sin x \cos x = 0$$

$$1 - 3 \tan^2 x - 2 \tan x = 0, \text{ on dividing both sides by } \cos^2 x$$

$$3 \tan^2 x + 2 \tan x - 1 = 0$$

$$\text{This is the same as (ii), so } x = 18.5^\circ, 135^\circ, 198.5^\circ, 315^\circ$$

(iv) $1 + \cos 2x = \cos x$.

$1 + 2\cos^2 x - 1 = \cos x$

$2\cos^2 x - \cos x = 0$ ¹

$\cos x(2\cos x - 1) = 0$

Solving $\cos x = 0$ gives $x = 90^\circ, 270^\circ$ ¹Solving $\cos x = \frac{1}{2}$ gives $x = 60^\circ, 300^\circ$ ¹

5. Differentiate

a) $y = x^2 \sec 3x, y' = 2x \sec 3x + 3x^2 \sec 3x \tan 3x$ ¹

b) $y = \frac{\sin x}{1 + \cos x}, y' = \frac{\cos x(1 + \cos x) + \sin^2 x}{(1 + \cos x)^2} = \frac{1 + \cos x}{(1 + \cos x)^2} = \frac{1}{1 + \cos x}$ ¹

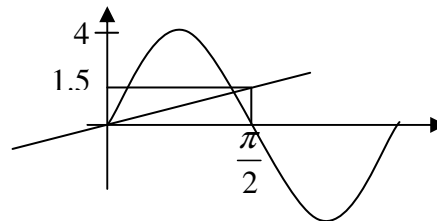
6. Evaluate

a) $\int x^2 \sin(x^3) dx = -\frac{1}{3} \cos(x^3) + C$ ¹

b) $\int_{\frac{\pi}{8}}^{\frac{\pi}{3}} \sec^2 2x dx = \frac{1}{2} \left[\tan 2x \right]_{\frac{\pi}{8}}^{\frac{\pi}{3}} = \frac{1}{2} (-\sqrt{3} - 1) = -\frac{\sqrt{3} + 1}{2}$ ¹

7. (i) Sketch the curves $y = 4 \sin 2x$ and $y = x$ for $0 \leq x \leq \pi$

1 mark for the curve, 1 for the line

(ii) Hence, solve $4 \sin 2x \geq x$, for $0 \leq x \leq \pi$, correct to 1 d.p.

$0 \leq x \leq 1.4$ ¹

(iii) Using the result of (ii) find the area enclosed by the curves $y = 4 \sin 2x$ and $y = x$ for $0 \leq x \leq \pi$.

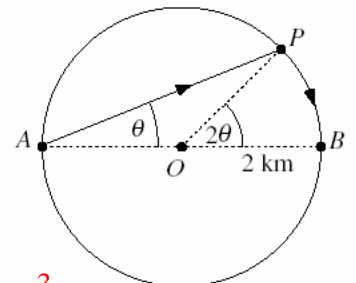
$$\int_0^{1.4} (4 \sin 2x - x) dx = \left[-2 \cos 2x - \frac{x^2}{2} \right]_0^{1.4} = 0.9 - (-2) = 2.9 \text{ units}^2$$
¹

8. The diagram shows a circular lake, centre O , of radius 2 km with diameter AB . Pat can row at 3 km/h and can walk at 4 km/h and wishes to travel from A to B as quickly as possible. Pat considers the strategy of rowing direct from A to a point P and then walking around the edge of the lake to B . Let $\angle PAB = \theta$ radians, and let the time taken for Pat to travel from A to B by this route be T hours.

(i) Show that $T = \frac{1}{3} (4 \cos \theta + 3\theta)$.

$$T = \frac{AP}{3} + \frac{PB}{4} \text{ where } \cos \theta = \frac{AP}{2}, \therefore AP = 2 \cos \theta \text{ and } PB = 2 \times 2\theta = 4\theta$$
¹

$$\therefore T = \frac{4 \cos \theta}{3} + \frac{4\theta}{4} = \frac{4 \cos \theta + 3\theta}{3}$$
¹

(ii) Find the value of θ for which $\frac{dT}{d\theta} = 0$. $\frac{dT}{d\theta} = \frac{1}{3} (-4 \sin \theta + 3) = 0$ when $\theta = \sin^{-1} \frac{3}{4}$ ¹(iii) To what point P , if any, should Pat row to minimise T ? Give reasons for your answer.

$$\frac{d^2T}{d\theta^2} = -\frac{4}{3} \cos \theta < 0 \text{ when } \theta = \sin^{-1} \frac{3}{4}, \therefore \text{this corresponds to a maximum time.}$$
¹

The time to walk along the arc AB is $\frac{2\pi}{4} = \frac{\pi}{2} \approx 1.6$ h, while the time to row from A to B is $\frac{4}{3} = 1.3$ h, so Pat should row directly to B to minimise the time.¹