Extension 1 – Test 2

1) Differentiate

(a) $y = x^2 \sin x$ (b) $y = \ln\left(\frac{2x+1}{x^2-1}\right)$ (c) $y = \frac{e^{3x}}{x^2}$ (d) $y = \frac{1}{\sqrt{1+4x^2}}$ 8

(a) $y' = 2x\sin x + x^2 \cos x$

(b) $y = \ln(2x+1) - \ln(x^2-1), y' = \frac{2}{2x-1} - \frac{2x}{x^2-1}$

(c) $y' = \frac{3e^{3x}x^2 - 2xe^{3x}}{x^4} = \frac{e^{3x}(3x-2)}{x^3}$

(d) $y' = -\frac{1}{2}\left(1+4x^2\right)^{-\frac{3}{2}}(8x) = -\frac{4x}{\sqrt{(1+4x^2)^3}}$

2) Find

(a) $\int \frac{\cos x}{\sqrt{1+\sin x}}dx$ (b) $\int \frac{2x}{3x+4}dx$ (c) $\int \frac{\ln 2x}{x}dx$ (d) $\int \frac{1}{3+4x^2}dx$ 8

(e) $\int \frac{x}{\sqrt{1-x}}dx$ using $x = 1-u^2$ (f) $\int \frac{dx}{x^2\sqrt{1-x^2}}$ using the substitution $x = \cos\theta$ 6

(a) Let $u = 1+\sin x, du = \cos x\, dx, \therefore \int \frac{du}{\sqrt{u}} = 2\sqrt{u} = 2\sqrt{1+\sin x} + C$

(b) $\int \frac{2x}{3x+4}dx = \frac{2}{3}\int \frac{3x+4-4}{3x+4}dx = \frac{2}{3}\int \left(1 - \frac{4}{3x+4}\right)dx = \frac{2}{3}\left(x - \frac{4}{3}\ln(3x+4)\right) = \frac{2x}{3} - \frac{8}{9}\ln(3x+4) + C$

(c) Let $u = \ln(2x), du = \frac{1}{x}dx, \therefore \int \ln 2x \frac{1}{x}dx = \int u\, du = \frac{u^2}{2} = \frac{(\ln 2x)^2}{2} + C$

(d) $\int \frac{dx}{(\sqrt{3})^2 + (2x)^2} = \frac{1}{2}\int \frac{2\,dx}{(\sqrt{3})^2 + (2x)^2} = \frac{1}{2} \times \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x}{\sqrt{3}} = \frac{1}{2\sqrt{3}}\tan^{-1}\frac{2x}{\sqrt{3}} + C$

(e) Let $x = 1-u^2, dx = -2u\, du, \therefore \int \frac{(1-u^2)(-2u\, du)}{\sqrt{1-(1-u^2)}} = -2\int(1-u^2)du = -2\left(u - \frac{u^3}{3}\right) = -2\sqrt{1-x} + \frac{2}{3}\sqrt{(1-x)^3} + C$

(f) Let $x = \cos\theta, dx = -\sin\theta\, d\theta, \therefore \int \frac{-\sin\theta\, d\theta}{\cos^2\theta\sqrt{1-\cos^2\theta}} = \int \frac{-\sin\theta\, d\theta}{\cos^2\theta \sin\theta} = -\int \sec^2\theta\, d\theta = -\tan\theta = -\frac{\sqrt{1-x^2}}{x} + C$

3) Find $\frac{d^2}{dx^2}\left(e^{x^2}\right)$. 4

$\frac{d}{dx}\left(e^{x^2}\right) = 2xe^{x^2}; \frac{d^2}{dx^2}\left(e^{x^2}\right) = \frac{d}{dx}\left(2xe^{x^2}\right) = 2e^{x^2} + 2x.2xe^{x^2} = 2e^{x^2}\left(1+2x^2\right)$

4) (a) Evaluate $\int_0^{\frac{\pi}{4}} \tan^2 x\, dx$. 3

$\int_0^{\frac{\pi}{4}} \tan^2 x\, dx = \int_0^{\frac{\pi}{4}} \left(\sec^2 x - 1\right)dx = \left[\tan x - x\right]_0^{\frac{\pi}{4}} = 1 - \frac{\pi}{4}$

(b) Evaluate $\int_0^1 x^3\sqrt{x^2+1}\, dx$ using $u = x^2+1$. 4

Let $u = x^2+1, du = 2x\, dx$. When $x = 0, u = 1$; when $x = 1, u = 2$.

$\int_0^1 x^2\sqrt{x^2+1}\, x\, dx = \frac{1}{2}\int_1^2 (u-1)\sqrt{u}du = \frac{1}{2}\int_1^2 \left(u^{3/2} - u^{1/2}\right)du = \left[\frac{u^{5/2}}{5} - \frac{u^{3/2}}{3}\right]_1^2 = \frac{4\sqrt{2}-1}{5} - \frac{2\sqrt{2}-1}{3} = \frac{2\sqrt{2}+2}{15}$

(c) Use the substitution $u = e^x + 1$, evaluate $\int_0^1 \frac{e^{2x}}{e^x+1}dx$. 4

Let $u = e^x + 1, du = e^x\, dx$. When $x = 0, u = 2$; when $x = 1, u = e+1$.

$\int_0^1 \frac{e^x}{e^x+1}e^x dx = \int_2^{e+1} \frac{(u-1)}{u}du = \int_2^{e+1}\left(1 - \frac{1}{u}\right)du = \left[u - \ln u\right]_2^{e+1} = e - 1 - \ln\frac{e+1}{2}$

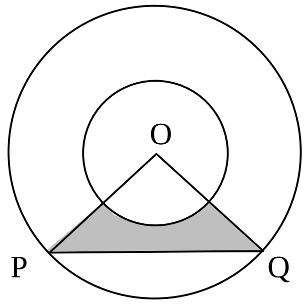
5) Differentiate $x\tan x$ with respect to x, hence, evaluate $\int_0^{\frac{\pi}{3}} x\sec^2 x\,dx$. 5

$\frac{d}{dx}(x\tan x) = \tan x + x\sec^2 x, \therefore \int_0^{\frac{\pi}{3}} \tan x\,dx + \int_0^{\frac{\pi}{3}} x\sec^2 x\,dx = \left[x\tan x\right]_0^{\frac{\pi}{3}}.$

but $\int_0^{\frac{\pi}{3}} \tan x\,dx = \int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x}dx = \left[-\ln(\cos x)\right]_0^{\frac{\pi}{3}} = -\ln\left(\cos\frac{\pi}{3}\right) - \ln(\cos 0) = -\ln\frac{1}{2} = \ln 2$

and $\left[x\tan x\right]_0^{\frac{\pi}{3}} = \frac{\pi}{3}\tan\frac{\pi}{3} - 0 = \frac{\pi\sqrt{3}}{3}.$

$\therefore \int_0^{\frac{\pi}{3}} x\sec^2 x\,dx = \frac{\pi\sqrt{3}}{3} - \ln 2.$



6) Two concentric circles have radii 2 cm and 4 cm and centre O. The points P and Q lie on the larger circle and ∠POQ = x, where $0 \le x \le \frac{\pi}{3}$.

(a) Prove that the area of the shaded region is $A = 8\sin x - 2x$. 2

Shaded area = Area of triangle − Area of sector $= \frac{1}{2}\times 4^2 \sin x - \frac{1}{2}\times 2^2 x = 8\sin x - 2x$

(b) Hence, find the value of x such that this area is maximum. 2

$\frac{dA}{dx} = 8\cos x - 2.$

$\frac{dA}{dx} = 0$ when $\cos x = \frac{1}{4}, \therefore x = 76°$. But $0 \le x \le 60°$ so max at $x = 60°$. $A_{\max} = 8\sin\frac{\pi}{3} - \frac{2\pi}{3} = 4\sqrt{3} - \frac{2\pi}{3}$ u^2

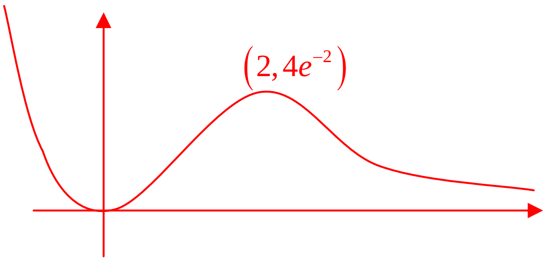
7) (a) If $y = x^2e^{-x}$. Find the coordinates of any turning points and determine their nature. 4

$y' = 2xe^{-x} - e^{-x}x^2 = e^{-x}x(2-x).$

$y' = 0$ when $x = 0, 2. \therefore$ Turning points $(0,0), (2, 4e^{-2})$

To determine the nature use the graph of $x(2-x)$, noting $e^{-x} > 0$, we will see that (0,0) is a minimum point while $(2, 4e^{-2})$ is a maximum point.

(b) Sketch the curve. 3



8) (a) PQ, where P is $(2p, p^2)$ and Q is $(2q, q^2)$, is a focal chord of the parabola $4y = x^2$. Prove that $pq = -1$. 3

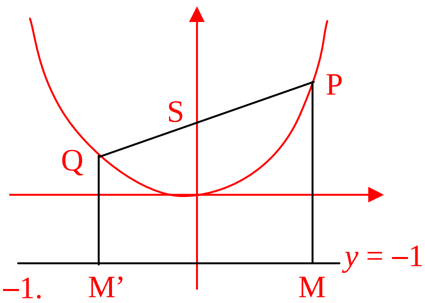
$m = \frac{p^2 - q^2}{2p - 2q} = \frac{p+q}{2}$

$y - p^2 = \frac{p+q}{2}(x - 2p)$

$2y - 2p^2 = (p+q)x - 2p^2 - 2pq$

$(p+q)x - 2y = 2pq$

Substitute the coordinates of the focus S(0,1) gives $-2 = 2pq, \therefore pq = -1.$



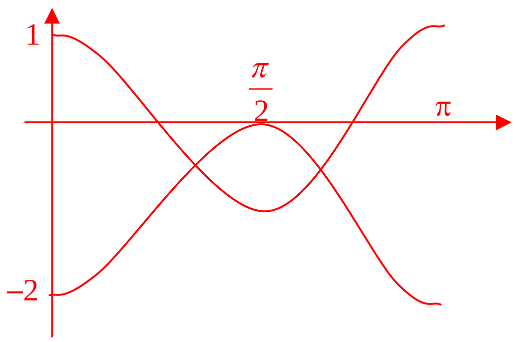
(b) Hence, find the length of PQ in terms of p in its simplest form. 3

$PS = PM = p^2 + 1, QS = QM' = q^2 + 1.$

$\therefore PQ = PS + QS = p^2 + q^2 + 2,\ \text{but } q^2 = \frac{1}{p^2}, \therefore PQ = p^2 + \frac{1}{p^2} + 2.$

9) (a) Sketch on the same axes the curves $y = \cos 2x$ and $y = -2\cos^2 x$, for $0 \le x \le \pi$. 3

$\cos 2x = 2\cos^2 x - 1, \therefore -2\cos^2 x = -(1 + \cos 2x)$: we'll draw $\cos 2x + 1$ then flip about the x-axis



(b) Find the x-coordinates of the points of intersection, for $0 \le x \le \pi$. 3

$\cos 2x = -2\cos^2 x = -(1 + \cos 2x)$

$2\cos 2x = -1$

$\cos 2x = -\frac{1}{2}$

$2x = \frac{2\pi}{3}, \frac{4\pi}{3}$

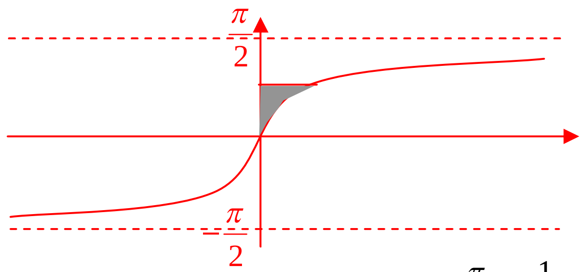
$x = \frac{\pi}{3}, \frac{2\pi}{3}$

(c) Find the area enclosed between the two curves, for $0 \le x \le \pi$.. 5

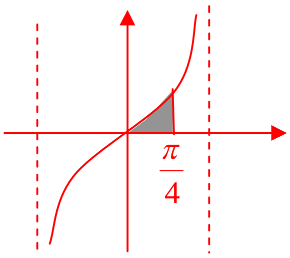
$\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \left|\cos 2x - 2\cos^2 x\right| dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \left|\cos 2x + (1 + \cos 2x)\right| dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \left|2\cos 2x + 1\right| dx = \left[\left|\sin 2x + x\right|\right]_{\frac{\pi}{3}}^{\frac{2\pi}{3}}$

$= \left|\left(-\frac{\sqrt{3}}{2} + \frac{2\pi}{3}\right) - \left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right)\right| = \left|\frac{\pi}{3} - \sqrt{3}\right| = \sqrt{3} - \frac{\pi}{3}\ \text{u}^2$

10) (a) Sketch the curve $y = \tan^{-1} 2x$. 2



(b) Show that the area obtained by this curve, the y-axis and the line $y = \frac{\pi}{4}$ is $\frac{1}{4}\ln 2$. 3



$f : y = \tan^{-1} 2x$

$f^{-1} : x = \tan^{-1} 2y, \therefore 2y = \tan x, \therefore y = \frac{1}{2}\tan x$

Required area $= \frac{1}{2}\int_0^{\frac{\pi}{4}} \tan x\, dx = \frac{1}{2}\left[-\ln(\cos x)\right]_0^{\frac{\pi}{4}} = -\frac{1}{2}\left(\ln\frac{1}{\sqrt{2}} + \ln 1\right) = \frac{1}{2}\ln\sqrt{2} = \frac{1}{4}\ln 2$

11) (a) Write $\sqrt{3}\cos x - \sin x$ in the form $R\cos(x+\alpha)$, hence, find the maximum value of $\sqrt{3}\cos x - \sin x$. 3

$R = \sqrt{(\sqrt{3})^2 + 1} = 2$

$$\sqrt{3}\cos x - \sin x = 2\left(\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x\right)$$
$$= 2\left(\cos x \cos\frac{\pi}{6} - \sin x \sin\frac{\pi}{6}\right)$$
$$= 2\cos\left(x + \frac{\pi}{6}\right). \therefore \text{ Maximum} = 2.$$

(b) What is the maximum value of $\left(\sqrt{3}+1\right)\cos x - \sin x$? 1

Maximum $= \sqrt{(\sqrt{3}+1)^2 + 1^2} = \sqrt{5 + 2\sqrt{3}}$

12) (a) Explain why $f(x) = x^2 + 2x + 5$ does not have an inverse function. 1

because $f(x)$ fails the horizontal line test

(b) Find the largest domain so that the restricted function may have an inverse function. 1

$f(x) = (x+1)^2 + 4, \therefore$ we can take $x \le -1$

(c) Find the inverse function in the restricted domain. 2

$f : y = (x+1)^2 + 4, x \le -1$

$f^{-1} : x = (y+1)^2 + 4$

$(y+1)^2 = x - 4$

$y + 1 = -\sqrt{x-4}$

$\therefore y = -1 - \sqrt{x-4}, y \le -1$

For part (b) I also accept $x \ge -1$, and consequently for part (c), $y = -1 + \sqrt{x-4}$

13) Find the exact values of (Working must be shown)

(a) $\cos^{-1}\left(\cos\frac{5\pi}{4}\right)$ (b) $\sin\left(\tan^{-1}\frac{9}{40}\right)$ (c) $\sin\left(2\cos^{-1}\frac{3}{5}\right)$ (d) $\sin^{-1}\frac{1}{2} - \sin^{-1}\frac{1}{3}$ 8

(a) $\cos^{-1}\left(\cos\frac{5\pi}{4}\right) = \cos^{-1}\left(\cos\left(2\pi - \frac{5\pi}{4}\right)\right) = \cos^{-1}\left(\cos\frac{3\pi}{4}\right) = \frac{3\pi}{4}$

(b) $\sin\left(\tan^{-1}\frac{9}{40}\right) = \sin\left(\sin^{-1}\frac{9}{41}\right) = \frac{9}{41}$ (see diagram (b))

(c) $\sin(2A) = 2\sin A\cos A = 2 \times \frac{4}{5} \times \frac{3}{5} = \frac{24}{25}$ (see diagram (c))

(d) $\sin(A-B) = \sin A\cos B - \cos A\sin B = \frac{1}{2} \times \frac{\sqrt{8}}{3} - \frac{\sqrt{3}}{2} \times \frac{1}{3} = \frac{\sqrt{8}-\sqrt{3}}{6}, \therefore \sin^{-1}\frac{1}{2} - \sin^{-1}\frac{1}{3} = \sin^{-1}\frac{\sqrt{8}-\sqrt{3}}{6}$ (see diagram (d).

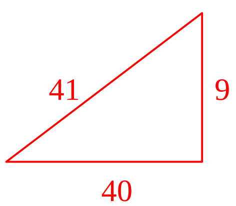


diagram (b)

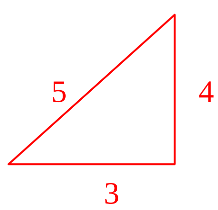


diagram (c)

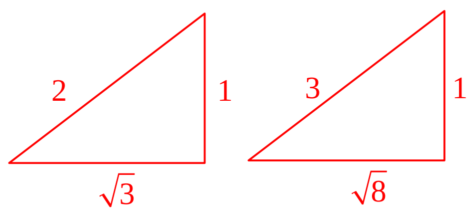


diagram (d)

14) (a) If all the letters of the word ENTERTAINMENT are mixed up and arranged in a row, in how many ways will neither of the three letters T's be together? 2

Arrange the non-T letters in $\frac{10!}{3!3!}$ ways then insert the three T's in 3 of the available 11 spaces (in front, back and between the letters) in ${}^{11}C_3$ ways, $\therefore \frac{10!}{3!3!} \times {}^{11}C_3$ ways.

(b) If only 5 letters are chosen, in how many ways does the selection contain more vowels than consonants? 3

The selection can have 4 vowels and 1 consonant or 3 vowels and 2 consonants, $\therefore\ {}^5C_4\,{}^8C_1 + {}^5C_3\,{}^8C_2$ ways.

15) Prove by mathematical induction that $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^n} > \frac{n}{2}$ for all n positive integers. 4

Let $n=1$, $\text{LHS}=1+\frac{1}{2}=\frac{3}{2}$, $\text{RHS}=\frac{1}{2}$, $\therefore$ LHS $>$ RHS

Assume $n=k$ is true, i.e. $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^k} > \frac{k}{2}$

Required to prove $n=k+1$ is true, i.e. $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^k}+\frac{1}{2^k+1}+\frac{1}{2^k+2}+\ldots+\frac{1}{2^k+2^k} > \frac{k+1}{2}$

Noting that, by assumption, $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^k} > \frac{k}{2}$

and that
$$\frac{1}{2^k}+\frac{1}{2^k+1}+\frac{1}{2^k+2}+\ldots+\frac{1}{2^k+2^k} > \frac{1}{2^k+2^k}+\frac{1}{2^k+2^k}+\frac{1}{2^k+2^k}+\ldots+\frac{1}{2^k+2^k}$$
$$= 2^k\left(\frac{1}{2^k+2^k}\right)$$
$$= 2^k\frac{1}{2\times 2^k}$$
$$= \frac{1}{2}$$

$$\therefore 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^k}+\frac{1}{2^k+1}+\frac{1}{2^k+2}+\ldots+\frac{1}{2^k+2^k} > \frac{k+1}{2}$$

$\therefore$ It's true for $n=k+1$

$\therefore$ It's true for all $n \geq 1$.