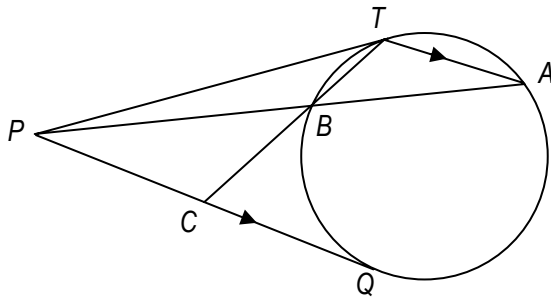


Practice Questions

Harder Extension 1 topics

(a)



(i) In $\triangle TPC$ and $\triangle PBC$,
 $\angle C$ is common,
 $\angle PTC = \angle BPC$ ($\angle PTC = \angle TAP$, angles in alternate segments are equal, and $\angle TAP = \angle BPC$, alternate angles on parallel lines)
 $\therefore \triangle TPC \sim \triangle PBC$ (equiangular)

(ii) $\frac{TC}{PC} = \frac{PC}{BC}$ (corresponding sides in similar triangles)

$\therefore PC^2 = TC \times BC$.

(iii) But $CQ^2 = CB \cdot CT$ (intersecting secants theorem)

$\therefore PC^2 = CQ^2, \therefore PC = CQ, \therefore C$ is the midpoint of PQ .

(b) The number of words that they are separated by at least two letters = Total – the number of words when they are together – the number of words when they are separated by 1 letter.

Total = $\frac{7!}{2!}$ words.

The two letters S 's are together in $6!$ words.

To separate the two S 's by one letter, choose one more letter from the other 5 letters

(5C1) to make a group, so together there are 4 distinct letters and a group, hence, $5!$ ways, but inside the group of the 3 letters the two S 's must not stay in the middle, \therefore only 1 way. $\therefore 5 \times 5!$ ways.

\therefore The two letters S 's are separated by at least two other letters in $\frac{7!}{2!} - 6! - 5 \times 5! = 1200$ ways.

$$(c) (i) f'(x) = \frac{\ln x - 1}{(\ln x)^2}.$$

For $x \geq 3$, $\ln 3 > 1$, $\therefore f'(x) > 0$, $\therefore f(x)$ is monotonic increasing.

$$(ii) \therefore \text{For } n \geq 3, \frac{n+1}{\ln(n+1)} > \frac{n}{\ln n}.$$

$$(n+1)\ln n > n\ln(n+1)$$

$$\ln n^{n+1} > \ln(n+1)^n$$

$\therefore n^{n+1} > (n+1)^n$, since $\ln x$ is increasing for all $x > 0$.

(d) (i) $(\cos x + i \sin x)^{2n+1} = \cos(2n+1)x + i \sin(2n+1)x$, by De Moivre's theorem.

$$\begin{aligned} (\cos x + i \sin x)^{2n+1} &= \binom{2n+1}{0} \cos^{2n+1} x + i \binom{2n+1}{1} \cos^{2n} x \sin x - \binom{2n+1}{2} \cos^{2n-1} x \sin^2 x \\ &\quad - i \binom{2n+1}{3} \cos^{2n-2} x \sin^3 x + \dots + i(-1)^n \binom{2n+1}{2n+1} \sin^{2n+1} x, \text{ by binomial expansion,} \end{aligned}$$

noting that the last term contains $i^{2n+1} = (i)^{2n} \cdot (i) = (-1)^n \cdot i$.

Equating the imaginary parts,

$$\sin(2n+1)x = \binom{2n+1}{1} \cos^{2n} x \sin x - \binom{2n+1}{3} \cos^{2n-2} x \sin^3 x + \dots + (-1)^n \binom{2n+1}{2n+1} \sin^{2n+1} x.$$

$$\therefore \frac{\sin(2n+1)x}{\sin^{2n+1} x} = \binom{2n+1}{1} \cot^{2n} x - \binom{2n+1}{3} \cot^{2n-2} x + \dots + (-1)^n.$$

(ii) Let $t = \cot^2 x$, the roots of the polynomial $p(t) = \binom{2n+1}{1} t^n - \binom{2n+1}{3} t^{n-1} + \dots + (-1)^n$

$= 0$ are the same as the roots of $\frac{\sin(2n+1)x}{\sin^{2n+1} x} = 0$.

Solving $\sin(2n+1)x = 0$ gives $x = \frac{k\pi}{2n+1}$, $k = 0, 1, 2, \dots, n$. These values of k are chosen so

that $0 \leq x \leq \frac{\pi}{2}$. But when $k = 0$, $\sin x = 0$, $\frac{\sin(2n+1)x}{\sin^{2n+1} x}$ is undefined, so $\frac{\sin(2n+1)x}{\sin^{2n+1} x} = 0$

only has n roots, $x = \frac{k\pi}{2n+1}$, $k = 1, 2, \dots, n$.

$\therefore p(t)$ has exactly n roots and they are $t = \cot^2 \left(\frac{k\pi}{2n+1} \right)$, $k = 1, 2, \dots, n$.

$$(iii) \sum_{k=1}^n \cot^2\left(\frac{k\pi}{2n+1}\right) = \sum \alpha = \frac{\binom{2n+1}{3}}{\binom{2n+1}{1}} = \frac{(2n+1)!}{3!(2n-2)!} = \frac{(2n)(2n-1)}{3!} = \frac{2n(2n-1)}{6}.$$

$$(iv) \operatorname{cosec}^2 x = \cot^2 x + 1, \therefore \sum_{k=1}^n \operatorname{cosec}^2\left(\frac{k\pi}{2n+1}\right) = \frac{2n(2n-1)}{6} + n = \frac{4n^2 + 4n}{6} = \frac{2n(2n+2)}{6}.$$

$$(v) \text{ Since } \cot x < \frac{1}{x} < \operatorname{cosec} x, \sum_{k=1}^n \cot^2\left(\frac{k\pi}{2n+1}\right) < \sum_{k=1}^n \left(\frac{2n+1}{k\pi}\right)^2 < \sum_{k=1}^n \operatorname{cosec}^2\left(\frac{k\pi}{2n+1}\right),$$

$$\frac{2n(2n-1)}{6} < \left(\frac{2n+1}{\pi}\right)^2 + \left(\frac{2n+1}{2\pi}\right)^2 + \dots + \left(\frac{2n+1}{n\pi}\right)^2 < \frac{2n(2n+2)}{6}.$$

$$\frac{2n(2n-1)}{6} \frac{\pi^2}{(2n+1)^2} < \frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{n^2} < \frac{2n(2n+2)}{6} \frac{\pi^2}{(2n+1)^2}.$$

$$\therefore \frac{\pi^2}{6} \left(\frac{2n}{2n+1}\right) \left(\frac{2n-1}{2n+1}\right) < \frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{n^2} < \frac{\pi^2}{6} \left(\frac{2n}{2n+1}\right) \left(\frac{2n+2}{2n+1}\right).$$

$$(vi) \text{ As } n \rightarrow \infty, \frac{2n}{2n+1} \rightarrow 1, \frac{2n-1}{2n+1} \rightarrow 1, \frac{2n+2}{2n+1} \rightarrow 1, \frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{n^2} \text{ is sandwiched}$$

$$\text{between two values which are the same, } \frac{\pi^2}{6}, \therefore \lim_{n \rightarrow \infty} \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}\right) = \frac{\pi^2}{6}.$$