

Practice Questions

Integration

$$(a) \int \frac{dx}{\sqrt{x(4-x)}} = \int \frac{dx}{\sqrt{4-(x-2)^2}} = \sin^{-1} \frac{x-2}{2} + C.$$

$$(b) \int \cos^3 x \sin^2 x \, dx = \int \cos x (1 - \sin^2 x) \sin^2 x \, dx = \int (\sin^2 x - \sin^4 x) \cos x \, dx \\ = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.$$

$$(c) \text{ Let } t = \tan \frac{x}{2}, dt = \frac{1}{2} \sec^2 \frac{x}{2} dx, \therefore dx = \frac{2dt}{\sec^2 \frac{x}{2}} = \frac{2dt}{1+t^2}.$$

$$\text{When } x=0, t=0; \text{ When } x=\frac{\pi}{4}, t=\tan \frac{\pi}{8} = \sqrt{2}-1. \quad (*)$$

$$\int_0^{\frac{\pi}{4}} \frac{\tan x}{1+\cos x - \sin x} dx = \int_0^{\sqrt{2}-1} \frac{\frac{2t}{1-t^2}}{1 + \frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2}} \frac{2dt}{1+t^2} \\ = \int_0^{\sqrt{2}-1} \frac{4t}{(1-t^2)(1+t^2+1-t^2-2t)} dt \\ = \int_0^{\sqrt{2}-1} \frac{2t}{(1-t^2)(1-t)} dt = \int_0^{\sqrt{2}-1} \frac{2t}{(1+t)(t-1)^2} dt \\ = \int_0^{\sqrt{2}-1} \left(\frac{-\frac{1}{2}}{1+t} + \frac{\frac{1}{2}}{t-1} + \frac{1}{(t-1)^2} \right) dt$$

$$\begin{aligned}
&= \left[-\frac{1}{2} \ln(1+t) + \frac{1}{2} \ln(t-1) - \frac{1}{t-1} \right]_0^{\sqrt{2}-1} \\
&= -\frac{1}{2} \ln \sqrt{2} + \frac{1}{2} \ln \frac{\sqrt{2}-2}{-1} - \left(\frac{1}{\sqrt{2}-2} - (-1) \right) \\
&= \frac{1}{2} \ln \frac{2-\sqrt{2}}{\sqrt{2}} - \left(\frac{\sqrt{2}+2}{-2} + 1 \right) \\
&= \frac{1}{2} \ln(\sqrt{2}-1) + \frac{\sqrt{2}}{2}.
\end{aligned}$$

(*) From $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$, let $x = \frac{\pi}{8}$ and $\tan \frac{\pi}{8} = t$, we have $1 = \frac{2t}{1-t^2}$, or $t^2 + 2t - 1 = 0$.

$$\therefore t = \frac{-2 \pm \sqrt{8}}{2} = -1 \pm \sqrt{2}.$$

As $\tan \frac{\pi}{8} > 0$, $\tan \frac{\pi}{8} = -1 + \sqrt{2}$.

(d) Let $u = \frac{\pi}{2} - x$, $du = -dx$

When $x = 0$, $u = \frac{\pi}{2}$; When $x = \frac{\pi}{2}$, $u = 0$.

$$\int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x} + e^{\cos x}} dx = \int_{\frac{\pi}{2}}^0 \frac{e^{\sin(\frac{\pi}{2}-u)}}{\sin(\frac{\pi}{2}-u) + e^{\cos(\frac{\pi}{2}-u)}} (-du) = \int_0^{\frac{\pi}{2}} \frac{e^{\cos u}}{e^{\cos u} + e^{\sin u}} du = \int_0^{\frac{\pi}{2}} \frac{e^{\cos x}}{e^{\sin x} + e^{\cos x}} dx$$

$$\text{Hence, } \int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x} + e^{\cos x}} dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{e^{\sin x} + e^{\cos x}}{e^{\sin x} + e^{\cos x}} dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{4}.$$

$$\begin{aligned}
\text{(e)} \quad \int_0^1 \frac{x^2 + 2}{(2x+1)(x^2+1)} dx &= \int_0^1 \left(\frac{\frac{9}{5}}{2x+1} + \frac{-\frac{2}{5}x + \frac{1}{5}}{x^2+1} \right) dx \\
&= \left[\frac{9}{10} \ln(2x+1) - \frac{1}{5} \ln(x^2+1) + \frac{1}{5} \tan^{-1} x \right]_0^1 \\
&= \frac{9}{10} \ln 3 - \frac{1}{5} \ln 2 + \frac{\pi}{20}.
\end{aligned}$$

(f) Let $u = x^n$, $du = nx^{n-1} dx$, $dv = \sin x$, $v = -\cos x$

$$\int_0^{\frac{\pi}{2}} x^n \sin x dx = \left[-x^n \cos x \right]_0^{\frac{\pi}{2}} + n \int_0^{\frac{\pi}{2}} x^{n-1} \cos x dx = n \int_0^{\frac{\pi}{2}} x^{n-1} \cos x dx$$

Let $u = x^{n-1}$, $du = (n-1)x^{n-2} dx$, $dv = \cos x$, $v = \sin x$

$$\therefore n \int_0^{\frac{\pi}{2}} x^{n-1} \cos x \, dx = n \left[x^{n-1} \sin x \right]_0^{\frac{\pi}{2}} - n(n-1) \int_0^{\frac{\pi}{2}} x^{n-2} \sin x \, dx$$

$$\therefore u_n = n \left(\frac{\pi}{2} \right)^{n-1} - n(n-1)u_{n-2}.$$

$$\text{Now, } u_6 = 6 \left(\frac{\pi}{2} \right)^5 - 30u_4,$$

$$u_4 = 4 \left(\frac{\pi}{2} \right)^3 - 12u_2,$$

$$u_2 = 2 \left(\frac{\pi}{2} \right) - 2u_0, \text{ where } u_0 = \int_0^{\frac{\pi}{2}} \sin x \, dx = \left[-\cos x \right]_0^{\frac{\pi}{2}} = 1$$

$$\therefore u_6 = \frac{3\pi^5}{16} - 30 \left[\frac{\pi^3}{2} - 12(\pi - 2) \right] = \frac{3\pi^5}{16} - 15\pi^3 + 360\pi - 720.$$